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Peridynamic Model of Poroelasticity based on Hamilton's Principle

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Peridynamic Model of Poroelasticity based on Hamilton's Principle

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Dedication

To my grandfather Binxiang who has always been watching me from heaven.

To my parents Jinghua and Xiaohong who have always been supporting and believing in me.

To my girlfriend Luying who has always been there for me through good and bad.

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Abstract

Peridynamic Model of Poroelasticity based on Hamilton's Principle

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Porous media theories play an important role in many branches of engineering. Despite significant advances, the existing theories suffer from many limitations and drawbacks when dealing with problems with discontinuities like fractures. The difficulties inherent in these problems arise from the basic incompatibility of spatial discontinuities with the partial differential equations that are used in the classical porous media theories. Peridynamics, a relatively new nonlocal formulation of continuum mechanics based on integral equations, provides a path forward in modeling spatial discontinuities in the field of solid mechanics. In this thesis, the nonlocal formulation of peridynamics is successfully combined with finite deformation poroelasticity. First, a thorough derivation of finite deformation poroelasticity based on

extended Hamilton's principle is conducted. Then we include the integral formulation of peridynamic theory when deriving the nonlocal momentum balance equations for poroelasticity once again using extended Hamilton's principle. To complete our nonlocal poroelasticity theory, we also develop a new class of peridynamic constitutive models. Finally, the correspondence of our peridynamic poroelasticity theory to the classical finite deformation poroelasticity theory is shown by demonstrating that our peridynamic equations can be reduced to the classical momentum balance equations for poroelasticity if smooth and homogeneous deformation is assumed.

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Chapter 1

Introduction

Geomechanics fracture modeling is of great interest and use in modern development of petroleum engineering. For example, hydraulic fracturing, a highly effective well stimulation technique, is very sensitive to how fractures propagate through deep-rock formations. It is too expensive or even impossible to conduct full-scale experiments on hydraulic fracturing in most casesr; so much of the stimulation design relies on numerical simulation. However, predictive numerical simulation of fracture growth in petroleum reservoirs still remains one of the biggest challenges in petroleum engineering.

There are two main reasons why classical numerical methods are not able to simulate fractures in underground formations. One is that when fractures are propagating during the simulation, the geometry of the formation that is being simulated will accordingly change with finite and discontinuous deformations. If classical numerical techniques are being used, this evolving geometry of the formation will then require that the mesh/grid of the simulation has to continuously change in order to fit the new geometry. In other words, one has to remesh the formation model every time step if they want to simulate growing fractures. The other reason is that the governing equations of the classical theories are local partial differential equations, involving spatial derivatives. The spatial derivatives needed for these partial differential equations are undefined on a crack tip or moving surface, and this basic incompatibility of cracks with the partial differential equations that are used in the classical theory will cause difficulties for accurate simulations [Macek and Silling, 2007]. For the past two decades, there have seen substantial work on numerical mod-

eling of fractures such as the extended finite element method (XFEM) and meshless methods [Zhuang et al., 2012] to solve difficulties encountered by classical numerical techniques, but most of these numerical methods are only striving to treat the numerical issues, while they are still based on local partial differential equations with spatial derivatives, leaving behind a mathematical inconsistency with the nature of cracks. Instead of using the XFEM or meshless methods, this work applies a newly developed nonlocal theory, peridynamics, to poroelasticity to derive a new nonlocal mathematical theory for modeling finite deformation poroelasticity.

Peridynamics is a recently developed theory of solid mechanics that replaces the partial differential equations of the classical continuum theory with nonlocal integral equations. Since the integral equations remain valid in the presence of displacement discontinuities like cracks, this method has the potential to model fractures with great generality and without complications of mathematical singularities that plague conventional continuum approaches [Macek and Silling, 2007]. It has already been shown that peridynamics has many advantages in the field of solid mechanics, but its application to modeling hydraulic fractures in petroleum engineering still needs to be further study.

In this thesis, a peridynamic model for finite deformation poroelasticity is developed. In Chapter 3, we treat the formation, a porous media, as a mixture containing two constituents, fluid and solid, using mixture theory; and apply extended Hamilton's principle to describing the motion of the binary mixture in the framework of continuum mechanics, which gives us a complete set of momentum balance equations for poroelasticity. Then our result is compared with Biot's theory and a new definition of effective stresses is introduced, which provides energy-level interpretations of Biot's effective stress and Biot's coefficient. In Chapter 4, we follow the same steps but without the constraints of classical continuum mechanics. We also use the extended Hamilton's principle to describe the motion of the binary mixture, but this time nonlocal integral expressions are used based on the ideas of peridynamics. As a result, a set of nonlocal integral momentum balance equations are obtained for poromechanics. In order to complete the nonlocal theory for poroelasticity, a new class of peridynamic constitutive models are developed in Chapter 5. In Chapter 6, the nonlocal theory is shown to reduce to classical momentum balance equations that are derived in Chapter 3, under the assumption that the deformation is homogeneous and smooth, which justifies our nonlocal theory as a

candidate model for describing finite deformation poroelasticity.

Chapter 2

Literature Review

2.1 Review of Poroelasticity

Porous media consists of a solid phase, several fluid phases and gas phases. The solid usually is referred to as a matrix or skeleton. Examples of porous materials are soils, rocks, porous aluminum foams, etc. The mechanics of porous media is of great relevance in many disciplines in science and engineering, such as geotechnical engineering, biomechanics, physical chemistry, agricultural engineering, and material science [Li et al., 2004]; therefore, there has been a large amount of research on developing the theories of porous media during the twentieth century and continuing to present day.

In twentieth century, progress was made towards creating a consistent porous media theory. It was Paul Fillunger who first pioneered the porous media theory of liquid-saturated porous solids in 1913 [Fillunger, 1913]. He investigated the uplift problem in saturated porous bodies. Then Karl von Terzaghi developed the one-dimensional consolidation theory, based on a variety of experimental data, and also an analogous procedure for heat propagation in 1934 [von Terzaghi and Rendulic, 1934]. He developed independently of Fillunger the concept of effective stress, which is of great importance in soil mechanics. In the time that followed, porous media theory was led by Maurice Biot who, in the main parts of his work, followed the scientific working of Terzaghi whose porous media theories were more or less developed intuitively only in parts founded on mechanical principles, but based on validation with experimental data [Biot, 1941, 1956]. Biot's theory had an immense

influence on the description of the mechanical behavior of saturated compressible media in recent decades [De Boer, 2012]. Even today, it still plays an important role in geomechanics. After the creation of the modern mixture theory in the 1960s, porous media theories got new impulses. Along with the volume fraction concept, modern porous media theory (mixture theory constrained by the volume fraction concept) developed [Renon and Prausnitz, 1968].

Biot generalized Terzaghi's theory of consolidation by extending it to the three-dimensional case, and by establishing equations valid for any arbitrary load varied with time [Biot, 1935, 1941]. He discussed the number of physical constants necessary to determine the properties of soil and developed the general equations for the prediction of settlement and stress for three-dimensional soil problems. Biot assumed the following properties of soil [Biot, 1941]:

- Isotropy
- Reversibility of stress-strain relations under final equilibrium conditions
- Linearity of stress-strain relations
- Small strains
- The water contained in the pores is incompressible
- The water may contain air bubbles
- The water flows through the porous skeleton according to Darcy's law

Based on these assumptions, Biot considered a small cube which is taken to be large enough compared to the size of the pores so that it can be treated as homogeneous, and at the same time small enough, compared to the scale of the macroscopic phenomen so that it can be considered as infinitesimal in the mathematical treatment. Then, he developed a model where the stresses in the soil are composed of two parts: one which is caused by the hydrostatic pressure of the water filling the pores, the other caused by the average stress in the skeleton. In this sense, the stress in the soil, $\mathbf{T}^s$, is carried partly by the water and partly by the solid constituent.

Because of the assumption of small strains in the soil, the strain tensor can be identified by linearized Green strain tensor $\mathbf{E}^L$

$$\mathbf{E}^L = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^\intercal],$$

where $\mathbf{u}$ is the displacement vector of the soil and ∇ is the gradient operator.

In order to describe completely the macroscopic conditions of the soil, Biot considered an additional variable representing the amount of water in the pores. He denoted the increment of water volume per unit volume of soil by θ, and called this quantity the variation in water content. Also, the increment of water pressure will be denoted by p. Since it is assumed that the changes in the soil occur by reversible processes the macroscopic condition of the soil must be a function of the stress and the water pressure. In the case that the strain and the variation in water content are small, the relation between these two sets of variables can be taken as linear in approximation. Additionally, the water pressure cannot produce any shear strain in the soil because of the assumed isotropy of the soil. Therefore, the relation between strain and stress can be written as [Biot, 1941]

$$\mathbf{E}^L = \frac{1}{2G}\mathbf{T}^s - \frac{\nu}{E}(\mathrm{tr}(\mathbf{T}^s) - \frac{p}{3H}\mathbf{I}), \tag{2.1}$$

where the constants E, G, and ν are Young's modulus, shear modulus, and Poisson's ratio of the solid skeleton, respectively; $\mathbf{I}$ is the second-order identity tensor; H is an additional physical constant. Biot further showed that the relationship of the water content θ with the soil stress $\mathbf{T}^s$ and water pressure p can be written as

$$\theta = \frac{1}{3H}\mathrm{tr}(\mathbf{T}^s) + \frac{p}{R},$$

where R is an another additional physical constant. Biot further solved equation (2.1) for T^s

$$\mathbf{T}^s = 2\mu\mathbf{E}^L + (\lambda\mathrm{tr}(\mathbf{E}^L)\cdot\mathbf{I} - np)\mathbf{I},$$

where μ and λ are the Lamé constants and n is defined by

$$n = \frac{K}{H},$$

where K is the bulk modulus. The water content, expressed in terms of the strains and the water pressure is then

$$\theta = n\mathrm{tr}(\mathbf{E}^L) + \frac{p}{Q},$$

with

$$\frac{1}{Q} = \frac{1}{R} - \frac{n}{H}.$$

The two elastic coefficients and the constants R and H completely define the physical parameters of an isotropic soil in equilibrium.

Biot's investigations later development the theory for propagation of stress waves in a porous elastic solid containing a compressible viscous fluid [Biot, 1956]. In 1957, Biot and Willis [1957] summarized the results and focused on the introduction of alternative variables, transverse isotropy, and elastic coefficients for incremental deformations of a prestressed material. Additionally, they present the constitutive equation for the total stress

$$\mathbf{T}^s + \mathbf{T}^f = -p(1 - \frac{\delta}{\kappa})\mathbf{I} + 2\mu\mathbf{E}_s + \lambda \mathrm{tr}(\mathbf{E}_s)\mathbf{I}, \tag{2.2}$$

where δ and κ characterize the compressibility of the real mineral solid and that of the porous solid skeleton, respectively. This equation describing the total stress in porous media is still of great use in petroleum engineering and geomechanics. Define the coefficient α as

$$\alpha := 1 - \frac{\delta}{\kappa}, \tag{2.3}$$

which is the well-known Biot's coefficient.

In addition to Biot's work, Frenkel's research concerning saturated soil is also of interest. He quantitatively determined the electric effect associated with the propagation of elastic vibrations in the soil [Frenkel, 1944]. Particularly, in relation to porous media theories, his basic equations of two-phase systems are still useful in poroelasticity.

Frenkel stated that theories of the motion of water in a soil based upon Darcy's law should take into account the fact that the particles of the soil could be elastically compressed and expanded. He assumed that the external forces and the hydrostatic pressure acted only on the liquid filling these pores. The macroscopic theory of soil only considers such distances that are large compared with the dimensions of the solid grains and/or the pores, and such elements of volume that contain a large number of these particles and pores. The porosity is defined as a coefficient ϕ equal to the ratio of the volume of the liquid-saturated pores v^f to the

total volume occupied by the soil, $v = v^s + v^f$. Then Frenkel defined the partial density ρ^α and the effective density $\rho^{\alpha R}$ of each phase. Referring the volumes v^α and v to unit mass

$$\rho^{\alpha R} = \frac{1}{v}^i, \qquad \rho^\alpha = \frac{1}{v},$$

where α could be s for solid and f for fluid. Note that the symbol α is already overloaded indicating both Biot's coefficient and the constituent index, but the intended use should be clear as the constituent index only appears as superscript. Consequently, with respect to the solid:

$$\rho^s = \rho^{sR}\frac{v^s}{v} = \rho^{sR}(1 - \frac{v^f}{v}) = \rho^{sR}(1 - n).$$

If all pores of the soil are completely filled with a liquid that can flow freely in and out of them, in order to remain in equilibrium and in the absence of external forces the liquid must be subjected to the same hydrostatic pressure p, at all points of the multiply connected space formed by the pores. The pressure p must be exerted on the solid skeleton of the soil.

Under the assumption of an absolutely rigid solid skeleton, the flow of the liquid phase is determined by Darcy's equation

$$\mathbf{v}_f = \frac{K^s}{\mu^f}(-\nabla p + \rho^f \mathbf{b}),$$

where $\mathbf{v}_f$ is the Darcy velocity, $\rho^f \mathbf{b}$ is the external force acting on the liquid contained in a unit volume of the soil, μ^f is the viscosity coefficient of the liquid, and K^s is the intrinsic permeability of the soil. Darcy's equation refers to steady flow. In the case of an unsteady flow, Frenkel showed that the flow equation is

$$\rho^f \frac{\partial \mathbf{v}_f}{\partial t} = -\phi\nabla p + \rho^f \mathbf{b} - \frac{\mu^f}{\kappa}\mathbf{v}_f,$$

where the coefficient κ is defined as

$$\kappa = \frac{K^s}{\phi}.$$

When considering the deformation of the solid skeleton, Frenkel obtained the final

form of the equations of motion for the liquid phase and solid phase:

$$\begin{aligned}\rho^f \frac{\partial \mathbf{v}_f}{\partial t} =& -\phi\nabla p + \rho^f \mathbf{b} - \frac{\mu^f}{\kappa}(\mathbf{v}_f - \mathbf{v}_s),\\ \rho^s \frac{\partial \mathbf{v}_s}{\partial t} =& \nabla\cdot\mathbf{T}^s - (1-\phi)\nabla p + \rho^s \mathbf{b} + \frac{\mu^f}{\kappa}(\mathbf{v}_f - \mathbf{v}_s).\end{aligned} \tag{2.4}$$

In 1972, Biot and Temple [1972] brought the mechanics of porous media to the same level of development as the classical theory of finite deformations in elasticity. Because the solid properties are transported and rotated with the motion, their theory is based on a Lagrangian or material decription of deformation, in contrast to the Eulerian or spatial description that is more appropriate for isotropic homogeneous fluids [Uzuoka and Borja, 2012]. Since then, a number of finite deformation theories for porous solids utilizing a Lagrangian formulation have emerged in the literature including Carter et al. [1979], Bedford and Drumheller [1979], Berryman and Thigpen [1985], Yatomi et al. [1989], Borja and Alarcón [1995], Armero [1999], and Larsson and Larsson [2002].

Among these fascinating works, the papers of Bedford and Drumheller [1979] and Berryman and Thigpen [1985] especially inspire some of the ideas of this thesis. Bedford and Drumheller presented a theory for treating a porous elastic material as a mixture which is based upon Hamilton's extended variational principle. By applying Hamilton's extended variational principle to the porous media as a mixture, there are several advantages [Bedford and Drumheller, 1979]:

- The kinematic information used is more extensive.
- The kinematic description which is used permits the volume fraction to be introduced explicitly.
- The kinematic description which is used permits the constitutive relations for the porous solid and for the fluid to depend only upon kinematic variables associated with the solid and the fluid respectively.
- The theory includes the kinetic energies of local expansion and contraction of the solid and of the fluid.

Instead of introducing constitutive relations for the volume fractions like many others have [Garg, 1971, Kenyon, 1978], Bedford and Drumheller presented the volume

fractions as independent kinematic variables which are obtained explicitly from the machinery of Hamilton's extended variational principle. Finally, they also showed that in the limit of large wavelength, their final equations are equivalent to the Biot equations.

Following the work of Bedford and Drumheller, Berryman and Thigpen [1985] also used the extended Hamilton's principle to derive the general equations of motion and they further included induced mass terms and microinertial terms in kinetic energy. They chose to use a semilinear approximation to express the internal energies in terms of densities and invariants of Lagrangian strain tensor

$$\begin{aligned}\Psi^s =&\frac{1}{2}aI_1^{s2} + bI_2^s + cI_1^s(\overline{\rho}^s - \overline{\rho}_0^s) + \frac{1}{2}d(\overline{\rho}^s - \overline{\rho}_0^s)^2 + eI_1^{s3} + fI_1^sI_2^s + gI_3^s \\ &+ mI_1^{s2}(\overline{\rho}^s - \overline{\rho}_0^s) + nI_2^s(\overline{\rho}^s - \overline{\rho}_0^s), \qquad (2.5)\\ \Psi^f =&\frac{1}{2}h(\overline{\rho}^f - \overline{\rho}_0^f)^2, \qquad (2.6)\end{aligned}$$

where Ψ^s and Ψ^f are the internal energy densities of solid and fluid; I_i^s $(i = 1, 2, 3)$ are the invariants of Lagrangian strain tensor for solid; $\bar{\rho}_0^s$ and $\bar{\rho}_0^f$ are the initial local densities of solid and fluid; $\bar{\rho}^s$ and $\bar{\rho}^f$ are the local densities of solid and fluid that vary throughout the motion; and the rest of coefficients are material constants.

Compared with Bedford and Drumheller's work, the semilinear approximations are more general in describing the internal energies. The first four terms of (5.30) are equivalent to the terms in the linear theory of Bedford and Drumheller while the new nonolinear terms are third order in terms of deformation gradient and density. Not only does this semilinear energy model raise the order of the approximation, but it is alos able to describe more coupling features of the material, e.g. the coupling between the density and the deviatoric stresses (the second invariants of the strain tensor) by including the cross terms $n\bar{I}_2^s(\bar{\rho}^s - \bar{\rho}_0^s)$ in equation (5.30) . These equations are shown explicitly here because that have important implications in the developments of this thesis.

2.2 Peridynamics

We now turn our attention toward the peridynamic theory of solid mechanics, where the terminology and notation of field of research is introduced in the following

sections.

2.2.1 Development of Peridynamics

The peridynamics theory of solid mechanics was first introduced by Silling [2000], as a reformulation of classical elasticity. Peridynamics attempts to unite the mathematical modeling of continuous media, evolving discontinuities, and particles within single framework, by replacing the spacial derivatives in partial differential equations of the classical momentum conservation with integral or integro-differential equations.

These integral equations, which give a mathematical consistency with the nature of displacement discontinuities like cracks in an otherwise continuously deforming body, are based on nonlocal models of internal forces within a body where material points interact with each other directly over finite distances. Peridynamic theory assumes that particles in a continuum interact with each other across finite distance, as in molecular dynamics Silling and Askari [2004].

The classical theory of solid mechanics is based on the assumption of a continuous distribution of mass within a deforming body. It further assumes that all internal forces are contact forces that act across zero distance. Based on these assumptions, the resulting mathematical description is in the form of partial differential equations which additionally require sufficient smoothness of the deformation and geometric entities in order for the PDEs to make sense. However, technology advancements and today's needs for accurate modeling have stretch the classical model of solid mechanics to it's limit of predictability. The ability to design and construct devices and materials at atomic length scales is only one challenge to the relevance of a theory that does not recognize the discrete nature of matter and the finiteness of distances across which forces can occur.

In typical engineering analysis, the elastic response of a heterogeneous material is treated by applying the classical solid mechanics equations with smoothed or homogenized material properties, which usually are measured using a laboratory test specimen much larger than any internal length scales in the material. This way of dealing with heterogeneous materials has been shown to fail to describe the mechanical response of the system as accurately as the nonlocal models such as peridynamics models which contain the information of micromechanical length

scale properties of the materials [Silling, 2014].

Such shortcomings also happen when it comes to evolving cracks and other discontinuities. The PDEs of the classical theory can not apply directly on a crack or dislocation since the deformation is discontinuous on these features, which has led to the need to treat cracks as pathological, rather than simply an aspect of deformation. All these considerations motivate the development of the peridynamic theory, which attempts to treat the evolution of discontinuities according to the same field equations as if the deformation were continuous.

The early research in peridynamics focused on fracture modeling, utilizing simple constitutive models. Silling and Askari [2005] first introduced a new meshfree method based on the peridynamic model of solid mechanics. In their work, damamge is incorporated in the theory at the level of two particles interactions, so localization and fracture occur as a natural outgrowth of the equation of motion and constitutive models. A numerical method for solving dynamic problems within the peridynamic theory was also described. First they take the peridynamic theory as a continuum version of molecular dynamics. The acceleration of any particle at $\mathbf{x}$ in the reference configuration at time t is found form

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{H}_\mathbf{x}} \mathbf{f}\left(\mathbf{u}(\mathbf{x}+\boldsymbol{\xi},t) - \mathbf{u}(\mathbf{x},t), \boldsymbol{\xi}\right)\ \mathrm{d}\boldsymbol{\xi} + \mathbf{b}(\mathbf{x},t), \tag{2.7}$$

where $\mathcal{H}_\mathbf{x}$ is a neighborhood of $\mathbf{x}$, $\mathbf{u}$ is the displacement vector field, $\mathbf{b}$ is a prescribed body force density field, ρ is mass density in the reference configuration, and $\mathbf{f}$ is a pairwise force function whose value is the force vector (per unit volume squared) that the particle $\mathbf{x}'$ exerts on the particle $\mathbf{x}$. The direct physical interaction (which occurs through unspecified means) between the particles at $\mathbf{x}$ and $\mathbf{x}'$ is called a *bond* which was treated in their paper as an elastic interaction, i.e. a spring, and this simple model is actually referred to as a bond-based model in later literature [Silling et al., 2007]. It is convenient to assume that for a given material there is a positive number δ, called the horizon, such that

$$\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi}) = 0 \quad \text{when} \quad |\boldsymbol{\xi}| > \delta,$$

where

$$\boldsymbol{\xi} = \mathbf{x}' - \mathbf{x}, \qquad \boldsymbol{\eta} = \mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t).$$

In other words, the particle $\mathbf{x}$ cannot "see" beyond this horizon. Then Silling and Askari [2005] further assumed, for simplicity, that the bond force $\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi})$ is in the direction of $\boldsymbol{\xi} + \boldsymbol{\eta}$, which leads to

$$\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{\boldsymbol{\xi} + \boldsymbol{\eta}}{|\boldsymbol{\xi} + \boldsymbol{\eta}|} f(\boldsymbol{\eta}, \boldsymbol{\xi}),$$

and the scalar bond force f depends only on the bond strech, defined by

$$s = \frac{|\boldsymbol{\xi} + \boldsymbol{\eta}|}{|\boldsymbol{\xi}|}.$$

They introduced a simple way to describe failure by allowing bonds to break when they are stretched beyond a predefined limit. After bond failure, there is no tensile force sustainable in the bond, and there is no provision for "healing" of a failed bond, which makes the model history dependent. To be specific, $f(\boldsymbol{\eta}, \boldsymbol{\xi})$ can be written as

$$f(\boldsymbol{\eta}, \boldsymbol{\xi}) = g(s)\mu(t, \boldsymbol{\xi}),$$

where g is the linear scalar-valued function given by

$$g(s) = cs,$$

where c is a constant and μ is a history-dependent scalar-valued function that takes on values of either 1 or 0:

$$\mu(t, \boldsymbol{\xi}) = \begin{cases} 1 & \text{if} \quad s(t', \boldsymbol{\xi}) < s_0 \quad \text{for all} \quad 0 \leq t' \leq t \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

where s_0 is the critical stretch for bond failure. Based on this definition of failure on bond level, it is unambiguous to introduce a local damage quantity at a point as

$$\varphi(\mathbf{x}, t) = 1 - \frac{\int_{\mathcal{H}_\mathbf{x}} \mu(\mathbf{x}, t, \boldsymbol{\xi}) \, \mathrm{d}V_{\boldsymbol{\xi}}}{\int_{\mathcal{H}_\mathbf{x}} \mathrm{d}V_{\boldsymbol{\xi}}},$$

where $\mathbf{x}$ is now included as an argument of μ as a reminder that it is a function of position in the body, and that $0 \leq \varphi \leq 1$, with 0 representing virgin material, while 1 representing complete disconnection of a point from all of the points with which

it initially interacted. The discretized form of the linearized peridynamic model assuming a homogeneous body is

$$\rho \ddot{u}_i^n = \sum_p C(x_p - x_i)(u_p^n - u_i^n) + b_i^n,$$

where C is a scalar-valued function. This solid peridynamic model was then classified as ordinary bond-based model [Silling et al., 2007].

A number of papers have investigated various aspects of the linear peridynamic theory. Silling et al. [2003] considered the static loading by body force density of an infinitely long, homogeneous bar. The results shows oscillations that decay at points remote from where the loading is applied which is a unique result of the nonlocality of peridynamic theory. Then Zimmermann [2005] further explored more features about linear peridynamic theory, including certain aspects of wave motion, material stability, and numerical solution techniques. Weckner and Abeyaratne [2005] studied the dynamics of the one-dimensional bar and obtain a Green's function representation of the solution. Finite element discretization techniques have also been studied by Weckner et al. [2009]. The convergence of the bond-based peridynamic theory to the equations of classical elasticity theory was also demonstrated by Zimmermann [2005], and in the context of isotropic linear elastic solid by Emmrich et al. [2007].

However, there are many restrictions of a theory based on pair interactions like the Possion's ratio in the theory restricted to be 1/4. These limitations motivated a rethinking of the bond-based peridynamic theory, which has led to a concept which preserves the idea of bonds carrying forces between pairs of particles, but allows for much more generality in material modeling. In the modern theory, the forces within each bond are not determined independently of each other. Instead, each bond force depends on the collective deformation, and/or rate of deformation, and history of all the bonds within the horizon of each material point [Silling and Lehoucq, 2010]. The resulting modified theory is called state-based, since the mathematical objects that convey information about the collective deformation of bonds are called peridynamic states. The earlier bond-based theory is a special case of state-based theory.

Silling et al. [2007] proposed a generalization of bond-based peridynamic theory with both ordinary and non-ordinary materials, and they called this generalized version of the peridynamic theory as the state-based theory. They included a spe-

cific isotropic material model where any Possion's ratio can be prescribed. They also have shown that any constitutive model from the classical theory can be adapted to the peridynamic theory using a nonlocal approximation to the deformation gradient tensor $\bar{\mathbf{F}}$.

$$\bar{\mathbf{F}}(\underline{\mathbf{Y}}) = \left(\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \underline{\mathbf{Y}}(\boldsymbol{\xi}) \otimes \boldsymbol{\xi} \mathrm{d}V_{\boldsymbol{\xi}} \right) \cdot \bar{\mathbf{K}}^{-1}, \tag{2.9}$$

where $\bar{\mathbf{K}}$ is the peridynamic shape tensor defined by

$$\bar{\mathbf{K}} = \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \boldsymbol{\xi} \otimes \boldsymbol{\xi} \mathrm{d}V_{\boldsymbol{\xi}}.$$

A peridynamic tensor was then derived by Silling and Lehoucq [2008], which has a mechanical interpretation similar to Piola stress tensor in the classical theory, providing the force per unit area across any imaginary internal surface. However, in the peridynamic case, the stress tensor is nonlocal. The forces involved are nonlocal forces in bonds that cross from one side of the surface to the other. The peridynamic operator for the internal force density can be expressed exactly as the divergence of the peridynamic stress tensor field. Thus the peridynamic equation of motion becomes formally the same as the classical equation [Silling and Lehoucq, 2010]. Ordinary material models has also been extended to include plasticity and viscoelastic behavior by Mitchell [2011].

Non-ordinary materials are of great importance because they allow for the development of material constitutive response models that have no analogue in the classical theories of elasticity. There have already been work done by O'Grady and Foster [2014] about non-ordinary state-based model for beams and flat shells. Additionally, there are works by Warren et al. [2009] and Breitenfeld et al. [2014] to apply non-ordinary models as a way to incorporate any classical stress tensor constitutive model into a computational code that solves the discrete peridynamic equations. These types of implementations are commonly called correspondence models in peridynamics because they derive from assuming a correspondence of strain-energy density functionals between the classical and peridynamic models.

Given the integral nature of the peridynamic conservation of linear momentum equation, an inherit length scale (usually called horizon) is present in the model. Bobaru and Hu [2012] studied the peridynamic horizon as an effective nonlocal in-

teraction that captures an effective length-scale or nonlocal behavior induced by the material microstructure and/or the type of dynamic loading. There is a influence function or kernel function inside the peridynamic integral, whose effect is also of interest. Many works done by Weckner and Silling [2011], Seleson and Parks [2011] and Silling [2014] have shown that influence functions or kernel functions can be used as a generalized way to localized influence from different physics present in a constitutive material model in order to recover interesting characteristics of material and geometric length scales.

Importantly, due to the nonlocal formulation, peridynamics can recover virtually all of both weakly and strongly nonlocal models presented in many literatures including Eringen and Edelen [1972] and Kunin [1982]. Most of these models are recovered by first restricting the kinematics to smooth fields such that the spatial derivatives can be evaluated or through linearization of the geometrically nonlinear finite deformation peridynamic theory.

2.2.2 Basic Concepts of Peridynamics

Peridynamic theory is a nonlocal theory describing the dynamics of a given body. Consider a continuum body which occupies the region $\mathcal{B}_0 \subset \mathbb{R}^3$ in the reference configuration at time $t = 0$ and the region $\mathcal{B} \subset \mathbb{R}^3$ at time t.

For completeness and in order to fix the notations, basic concepts and notations related to peridynamics theory are briefly review in this section, primarily following the notation and approach used in Silling et al. [2007] and Silling and Lehoucq [2010].

Peridynamic States

For a given material point in the reference configuration $\mathbf{X} \in \mathcal{B}_0$, let $\mathcal{N}$ be the neighborhood of radius δ with center $\mathbf{X}$. Define the family of $\mathbf{X}$ by

$$\mathcal{H}(\mathbf{X}) = \{\boldsymbol{\xi} \in \mathbb{R}^3 | \boldsymbol{\xi} + \mathbf{X} \in \mathcal{N} \cap \mathcal{B}_0\},$$

and the positive number δ is called the horizon, providing a physical length scale. Consider anothe material point $\mathbf{Q}$ inside the famliy $\mathcal{H}(\mathbf{X})$. From the perspective of

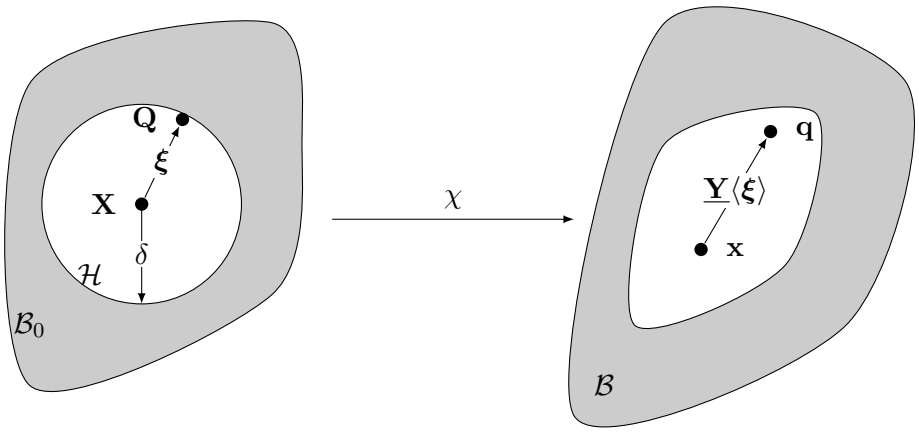

Figure 2.1: Schematic of peridynamic body with a horizon

point $\mathbf{X}$, the bond to point $\mathbf{Q}$ is the vector

$$\boldsymbol{\xi} := \mathbf{Q} - \mathbf{X}.$$

Through the processes of deformations and dynamic movements, the configuration of the body $\mathcal{B}_0$ becomes the current configuration (deformed configuration) $\mathcal{B}$ at time t. Let χ denote the deformation one-to-one mapping between the reference and current configuration: $\mathcal{B}_0 \to \mathcal{B}$, and points $\mathbf{X}$ and $\mathbf{Q}$ map to

$$\mathbf{x} = \chi(\mathbf{X}), \qquad \mathbf{q} = \chi(\mathbf{Q}),$$

respectively. A schematic description of a peridynamic body and the family at $\mathbf{X}$ is shown in Figure 2.1.

A peridynamic state $\underline{A}\langle\cdot\rangle$ is a function defined on $\mathcal{H}$. The angle brackets $\langle\cdot\rangle$ are used to identify the bond vector the state operates on, and parentheses are used to indicated dependencies of the state on other quantities including space $\mathbf{X}$ and time t. Note that the field dependence on $\mathbf{X}$ and t is often suppressed to simplify notations. If the value of $\underline{A}(\mathbf{X})\langle\boldsymbol{\xi}\rangle$ is a scalar, then it is scalar-state. If the value of $\underline{\mathbf{A}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle$ is a vector, then it is a vector-state [Silling and Lehoucq, 2010].

$$\underline{A}(\mathbf{X})\langle\boldsymbol{\xi}\rangle : \mathcal{H}(\mathbf{X}) \to \mathbb{R},$$

$$\underline{\mathbf{A}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle : \mathcal{H}(\mathbf{X}) \to \mathbb{R}^3.$$

Let $\mathcal{S}(\mathbf{X})$ denote the set of all scalar-states and $\mathcal{V}(\mathbf{X})$ denote the set of all vector-state at point $\mathbf{X}$.

$$\mathcal{S}(\mathbf{X}) = \mathcal{H}(\mathbf{X}) \times \mathbb{R},$$

$$\mathcal{V}(\mathbf{X}) = \mathcal{H}(\mathbf{X}) \times \mathbb{R}^3.$$

In order to simplify the notation, it is convenient to define the dot products between two states

$$\underline{a} \cdot \underline{b} = \int_{\mathcal{H}} \underline{a}\langle\boldsymbol{\xi}\rangle \underline{b}\langle\boldsymbol{\xi}\rangle \, \mathrm{d}\boldsymbol{\xi},$$

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, \mathrm{d}\boldsymbol{\xi}.$$

where $\underline{a}$ and $\underline{b}$ are scalar-states, $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are vector-states, the operation $\cdot$ in the integral denotes the standard inner-product between vectors in $\mathbb{R}^3$, and $\mathrm{d}\boldsymbol{\xi}$ is another notation for $\mathrm{d}V_{\boldsymbol{\xi}}$, indicating that the volume integral is done in the reference configuration varying with the bond $\boldsymbol{\xi}$. It is also useful to define a scalar-state $|\underline{\mathbf{A}}|$ describing the length of the vector-state $\underline{\mathbf{A}}$ by

$$|\underline{\mathbf{A}}|\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle|.$$

Most of the constitutive models in peridynamics involve functions of states, and it is helpful to define a notation for derivatives of such functions, or so-called their Fréchet derivatives. If $\psi(\cdot) : \mathcal{S} \to \mathbb{R}$ is a function of a scalar-state, its Fréchet derivative $\underline{\nabla}\psi$, if it exists, is defined as

$$\psi(\underline{A} + \underline{a}) = \psi(\underline{A}) + \underline{\nabla}\psi(\underline{A}) \bullet \underline{a} + o(\underline{a}), \tag{2.10}$$

for all scalar-states $\underline{A}$ and $\underline{a}$. $\underline{\nabla}\psi$ is a scalar-state. If $\Psi(\cdot)$ is a function of a vector-state, its Fréchet derivative $\underline{\nabla}\Psi$, if it exists, is similarly defined as

$$\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) = \Psi(\underline{\mathbf{A}}) + \underline{\nabla}\Psi(\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}} + o(|\underline{\mathbf{a}}|), \tag{2.11}$$

for all vector-states $\underline{\mathbf{A}}$ and $\underline{\mathbf{a}}$. $\underline{\nabla}\Psi$ is a vector-state.

An important example of vector-states is the deformation vector-state $\underline{\mathbf{Y}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle \in$

$\mathcal{V}(\mathbf{X})$, and it is defined as

$$\underline{\mathbf{Y}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle := \mathbf{q} - \mathbf{x} = \chi(\mathbf{Q}) - \chi(\mathbf{X}), \qquad \forall \boldsymbol{\xi} \in \mathcal{H}(\mathbf{X}).$$

The deformation vector-state is the primary deformation measure in peridynamics because it assigns every bond in the reference configuration to its deformed image in the current configuration. A critical, physically motivated, kinematic constraint in continuum mechanics is that distinct material points in the reference configuration remain distinct in the deformed configuration, or in other words, the deformation mapping must be one-to-one [Tupek, 2014]. Written in terms of peridynamic deformation state, it is stated as:

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{0} \qquad \text{if and only if} \qquad \boldsymbol{\xi} = \mathbf{0},$$

which means that two distinct particles never occupy the same point as the deformation progresses.

The peridynamic equations of motion are generally written in terms of the force vector-state $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$, which is a bond-force acting as an interaction force between nearby materials points. In peridynamics, this force vector-state $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$ is very similar to the stress tensor in the classical continuum theories, as the general peridynamic equation of motion can be written as [Silling et al., 2007]:

$$\rho_0 \ddot{\mathbf{y}} = \int_{\mathcal{H}} \underline{\mathbf{T}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}(\mathbf{X}')\langle -\boldsymbol{\xi}\rangle \, \mathrm{d}\boldsymbol{\xi} + \mathbf{b}, \tag{2.12}$$

where $\mathbf{b}$ is an externally applied body force, and ρ_0 is the reference material density. Note that this equation is actually expressing conservation of linear momentum. Angular momentum is conserved provided that the material is non-polar and the following constitutive condition holds (S. A. Silling et al,. 2007):

$$\int_{\mathcal{H}} \underline{\mathbf{T}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle \, \mathrm{d}\boldsymbol{\xi} = \mathbf{0}, \qquad \forall \mathbf{X} \in \mathcal{B}_0 \tag{2.13}$$

The essential differentiator of peridynamics from classical continuum mechanics is that the theory is inherently nonlocal, in the sense that material points interact through long-range forces represented by the force vector-state $\underline{\mathbf{T}}(\mathbf{X})\langle\boldsymbol{\xi}\rangle$, whose bond-wise components can be loosely interpreted as a force per unit refer-

ence volume-squared at point $\mathbf{X}$ due to interactions with the point $\mathbf{Q}$ [Tupek, 2014].

Constitutive Modeling in Peridynamics

A constitutive model in the state-based peridynamic theory is a relation that provides values for the force vector state field in terms of the deformation vector state field and possibly other variables as well. The general form of a constitutive model will be written as [Silling et al., 2007]

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \Lambda),$$

where $\hat{\underline{\mathbf{T}}} : \mathcal{V} \to \mathcal{V}$ is bounded and Riemann-integrable on $\mathcal{H}$, and Λ denotes all variables other than the current deformation vector-state that $\underline{\mathbf{T}}$ may depend on for some particular material. For example, if the material is nonhomogeneous and also rate dependent, the constitutive model would additionally depend on space position $\mathbf{X}$ and the time derivative of the deformation state:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \dot{\underline{\mathbf{Y}}}, \mathbf{X}).$$

Ordinary and Bond-based Models A constitutive peridynamic model $\hat{\underline{\mathbf{T}}}$ is called ordinary if every force vector-state is parallel to the deformation vector-state, or expressed using peridynamic vector-states (as illustrated in Fig. 2.2)

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{0}, \qquad \forall \boldsymbol{\xi} \in \mathcal{H}. \tag{2.14}$$

Comparing this condition with (2.13) , it is straightforward to see that ordinary peridynamic models automatically satisfy the angular momentum balance. For elastic materials, this condition is equivalent to requiring the elastic energy only depends on the distance between neighboring material points in the current configuration regardless of the angles between deformed material points.

A material model is called bond-based, if each bond has its own constitutive relation, independent of the others. In other words, there is a function $\hat{\mathbf{t}}(.,.)$ on $\mathbb{R}^3 \times \mathcal{H}$ such that

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \hat{t}(\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi}), \qquad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

The requirement for angular momentum balance within a force state as equation

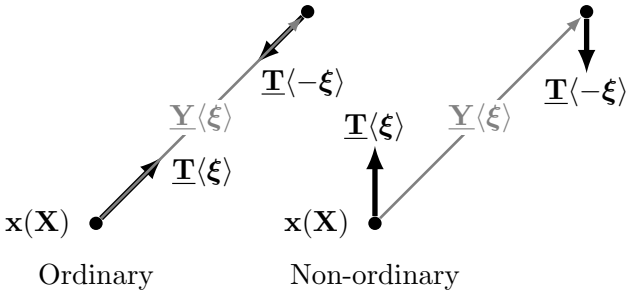

Figure 2.2: Ordinary vs. non-ordinary materials

(2.13) implies that the bond-based material model is ordinary [Silling et al., 2007].

Generally, ordinary peridynamic models are formulated in terms of a so-called extension scalar-state $\underline{e}\langle\boldsymbol{\xi}\rangle$ defined as

$$\underline{e}\langle\boldsymbol{\xi}\rangle := |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|,$$

which measure the change in distance between material points relative to their initial separation distance in the reference configuration. The most common example of an ordinary peridynamic constitutive model, so-called a linear ordinary elastic solid, has a strain energy density of the form [Silling et al., 2007]:

$$\psi(\underline{\mathbf{Y}}) = \frac{k\theta^2}{2} + \frac{\alpha}{2}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{e}^d\langle\boldsymbol{\xi}\rangle \cdot \underline{e}^d\langle\boldsymbol{\xi}\rangle \,\mathrm{d}\boldsymbol{\xi}, \tag{2.15}$$

where $\underline{\omega}\langle\boldsymbol{\xi}\rangle$ is the influence function; k is the material bulk modulus; α is a material property representing the materials resistance to shearing; θ is the volumetric dilatation defined as

$$\theta := \frac{3}{\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi}\cdot\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{\xi}} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{e}\langle\boldsymbol{\xi}\rangle|\boldsymbol{\xi}|\,\mathrm{d}\boldsymbol{\xi},$$

and $\underline{e}^d$ is the deviatoric extension scalar-state defined as

$$\underline{e}^d := \underline{e}\langle\boldsymbol{\xi}\rangle - \frac{\theta|\boldsymbol{\xi}|}{3}.$$

Constitutive Correspondence Models Given that adequate material models based on classical theory are often already well-developed, it could be really helpful

for peridynamic's application to those situations if there is a state-based peridynamic constitutive model that results in the same physical properties in those situations where a meaningful comparison between the two can be made. With this purpose in mind, Silling first introduced a constitutive correspondence peridynamic model in 2007. He developed a link between peridynamic models with most of classical models by approximating the deformation gradient in the classical theories using his matrix-state peridynamic deformation gradient $\bar{\mathbf{F}} \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$\bar{\mathbf{F}}(\underline{\mathbf{Y}}) := \left[\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle (\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \right] \cdot \bar{\mathbf{K}}^{-1}, \tag{2.16}$$

where $\underline{\omega}$ is a scalar state called influence function which must satisfy

$$\underline{\omega}\langle \boldsymbol{\xi} \rangle > 0, \qquad \forall \boldsymbol{\xi} \in \mathcal{H},$$

and $\bar{\mathbf{K}} \in \mathbb{R}^3 \times \mathbb{R}^3$ is also a matrix-state called the shape tensor:

$$\bar{\mathbf{K}} := \int_{\mathcal{H}} \underline{\omega}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}. \tag{2.17}$$

It has been shown that this peridynamic deformation gradient $\bar{\mathbf{F}}$ defined in this way is identical to the classical deformation gradient under the condition that the deformation is homogeneous [Silling et al., 2007]. The deformation vector-state $\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle$ represents a much richer way of describing how a body deforms than the classical idea of a deformation gradient tensor $\mathbf{F}$, because the deformation vector-state $\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle$ contains all the deformed configurations of every bond inside $\mathcal{H}$. However, we can still use this peridynamic deformation gradient $\bar{\mathbf{F}}$ as a tool to adapt classical material models in peridynamic theories. To be specific, suppose a random material (in the sense of classical elasticity theory) has a given strain energy density function $\hat{\psi}(\mathbf{F})$, where $\mathbf{F}$ is the classical deformation gradient tensor. We can then directly apply this material model in the state-based peridynamic framework, simply by replacing the classical deformation gradient $\mathbf{F}$ with our peridynamic deformation tensor $\bar{\mathbf{F}}$:

$$\psi(\underline{\mathbf{Y}}) = \hat{\psi}(\mathbf{F}(\bar{\underline{\mathbf{Y}}})),$$

where $\psi(\underline{\mathbf{Y}})$ is the strain energy density in the framework of state-based peridynamic.

Chapter 3

Finite Deformation Poroelasticity Equations from Hamilton's Principle

In this chapter, we follow the variational method used by Bedford and Drumheller [1979], and apply Hamilton principle within mixture theory to poroelasticity assuming finite deformation in a more complete and rigorous way. The first part of the chapter focus on introducing the basic concepts and notations of continuum mechanics in mixture theory, and then based on these concepts, momentum equations for a porous media are derived using extended Hamilton's principle. Since the derivation is done in the framework of finite deformation, the result equations' correspondence to the classical Biot's equations is shown in the next part of the chapter. Finally, we illustrate the relation between the Piola-Kirchhoff stress term in our equations and Biot's effective stress and then present an insightful interpretation on effective stress.

3.1 Basic Concepts of Continuum Mechanics in Mixture Theory

Consider a continuum body which is an immiscible mixture of two constituents (solid and fluid). According to mixture theory, these two constituents simultaneously

occupy a same material point identified by a position vector $\mathbf{X}$ in the reference configuration of a body $\mathcal{B}_0$. Generally, the two constituents, solid and fluid, can move relative to one another each described by their own position vector $\mathbf{x}^s$ and $\mathbf{x}^f$ in the current (deformed) configuration of the body $\mathcal{B}$. Let χ^s and χ^f denote the mapping between the reference and current configuration for solid and fluid respectively and assume that χ^s and χ^f are sufficiently smooth and one-to-one mappings. Based on these definitions, we can write

$$\mathbf{x}^\alpha = \chi^\alpha(\mathbf{X}, t), \qquad \alpha = s, f$$

where the superscription s and f represent the solid and fluid respectively. From continuum mechanics, the deformation gradient $\mathbf{F}^\alpha$ for each constituent is then defined as

$$\mathbf{F}^\alpha = \frac{\partial \chi^\alpha}{\partial \mathbf{X}}.$$

Following arguments from vector calculus, the infinitesimal volume occupied by the constituents in the reference configuration $\mathrm{d}V_0$ is related to the infinitesimal volume of the α constituent in the current configuration $\mathrm{d}V^\alpha$ through the Jacobian determinant J^α:

$$\mathrm{d}V^\alpha = J^\alpha \, \mathrm{d}V_0, \tag{3.1}$$

where the Jacobian determinant J^α is given by

$$J^\alpha = \det(\mathbf{F}^\alpha) = \det\left(\frac{\partial \chi^\alpha}{\partial \mathbf{X}}\right).$$

Assume that both constituents occupy an identical volume $\mathcal{B}_0$ with the same surface $\partial\mathcal{B}_0$ at time t_1 and another identical volume $\mathcal{B}$ with the same surface $\partial\mathcal{B}$ at time t_2, or, in other words, they do not separate at the boundaries but are allowed to move relative to one another internally, over a time interval $[t_1, t_2]$. Even if it is completely possible that some part of the volume $\mathcal{B}$ at time t_2 is fully taken by one constituent with only a minimum amount of the other constituent, we will say both constituents still simultaneously occupy the material while just letting the volume fraction (which will be mentioned later) of the one constituent to be really small.

Then, the mass conservation of the body from time t_1 to t_2 can be expressed as:

$$\int_{\mathcal{B}_0} \rho_0^\alpha \, \mathrm{d}V_0 = \int_{\mathcal{B}} \rho^\alpha \, \mathrm{d}V^\alpha,$$

where ρ^α is the partial density defined as the mass of α constituent divided by the total volume of all constituents in the current configuration. The subscript 0 is used to indicate evaluation in the reference configuration. Substitute equation (3.1) , we have

$$\int_{\mathcal{B}_0} \rho_0^\alpha \, \mathrm{d}V_0 = \int_{\mathcal{B}} \rho^\alpha J^\alpha \, \mathrm{d}V_0. \tag{3.2}$$

Equation (3.2) must hold for any arbitrary volume, therefore we can conclude with

$$J^\alpha = \frac{\rho_0^\alpha}{\rho^\alpha}.$$

However, the partial density does not necessarily reflect the real value of the constituent's actual density since it is mass per unit total volume. It is very useful to define a local density for each constituent to accomplish that. The local densities $\bar{\rho}^\alpha$, which is defined as the mass of the α constituent divided by the volume of the α constituent in the current configuration, are related to the partial densities ρ^α by

$$\rho^\alpha = \phi^\alpha \bar{\rho}^\alpha,$$

where ϕ^α is the volume fraction of the α constituent which is defined as the volume of the α constituent over the total volume at any given material point. Note that the definition of volume fraction gives us an equation called the *volume fraction constraint*

$$\sum_{\alpha=s,f} \phi^\alpha = 1. \tag{3.3}$$

Along with these definitions of partial densities and volume fractions, our mass conservation equation can be further written as

$$J^\alpha = \frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha}. \tag{3.4}$$

3.2 Conservation of Momentum Derivation using Extended Hamilton's Principle

The extended Hamilton's principle states that among admissible motions, the actual motion of a body is such that

$$\int_{t_1}^{t_2} \delta(T - V) + \delta W + \sum_k \delta C_k \mathrm{d}t = 0, \tag{3.5}$$

where δ is the variation operator from the calculus of variation; T is the kinetic energy of the body; V is the potential energy; W is the virtual work done by external loads on the body; C_k are the constraints set on the body's motion; and t_1, t_2 are the fixed times in which the configuration of the body is assumed to be prescribed.

We first focus on the constraints imposed on the porous media. As discussed in the previous section, there are two constraints: one is the mass conservation equations (3.4) , the other is the volume fraction constraint (3.3) , denoted by C_1 and C_2 respectively:

$$C_1 = \sum_{\alpha=s,f} \int_{\mathcal{B}_0} \lambda^\alpha \left(J^\alpha - \frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha} \right) \mathrm{d}V_0, \tag{3.6}$$

$$C_2 = \int_{\mathcal{B}} p \left(\sum_{\alpha=s,f} \phi^\alpha - 1 \right) \mathrm{d}V, \tag{3.7}$$

where λ^α and p are Lagrange multipliers. Note that there is a significant difference between the Lagrange multipliers λ^α and p. $\lambda^\alpha(\mathbf{X})$ is defined on the reference configuration because the constraint (mass conservation) is expressed in the reference configuration while $p(\mathbf{x})$ is defined in the current configuration given that the volume fraction constraint must be satisfied at every material point in the current configuration.

Taking the first variation of equation (3.6) using $\mathbf{x}^\alpha$, ϕ^α, and $\bar{\rho}^\alpha$ as inde-

pendent fields gives

$$\begin{aligned}
\delta C_1 &= \sum_{\alpha=s,f} \int_{\mathcal{B}_0} \lambda^\alpha \, \delta J^\alpha - \lambda^\alpha \, \delta \left(\frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha} \right) \, \mathrm{d}V_0, \\
&= \sum_{\alpha=s,f} \int_{\mathcal{B}_0} \lambda^\alpha J^\alpha (F_{ji}^\alpha)^{-1} \frac{\partial \delta x_i^\alpha}{\partial X_j} + \lambda^\alpha J^\alpha \left(\frac{\delta \phi^\alpha}{\phi^\alpha} + \frac{\delta \bar{\rho}^\alpha}{\bar{\rho}^\alpha} \right) \, \mathrm{d}V_0, \\
&= \sum_{\alpha=s,f} \int_{\partial \mathcal{B}_0} \lambda^\alpha J^\alpha N_j (F_{ji}^\alpha)^{-1} \delta x_i^\alpha \mathrm{d}S_0 \\
&\quad - \int_{\mathcal{B}_0} \frac{\partial \left(\lambda^\alpha J^\alpha (F_{ji}^\alpha)^{-1} \right)}{\partial X_j} \delta x_i^\alpha + \lambda^\alpha J^\alpha \left(\frac{\delta \phi^\alpha}{\phi^\alpha} + \frac{\delta \bar{\rho}^\alpha}{\bar{\rho}^\alpha} \right) \, \mathrm{d}V_0, \\
&= \sum_{\alpha=s,f} \int_{\partial \mathcal{B}_0} \lambda^\alpha J^\alpha \mathbf{N} (\mathbf{F}^\alpha)^{-\intercal} \cdot \delta \mathbf{x}_i^\alpha \mathrm{d}S_0 \\
&\quad - \int_{\mathcal{B}_0} \nabla \cdot \left(\lambda^\alpha J^\alpha (\mathbf{F}^\alpha)^{-\intercal} \right) \cdot \delta \mathbf{x}^\alpha + \lambda^\alpha J^\alpha \left(\frac{\delta \phi^\alpha}{\phi^\alpha} + \frac{\delta \bar{\rho}^\alpha}{\bar{\rho}^\alpha} \right) \, \mathrm{d}V_0,
\end{aligned} \tag{3.8}$$

where integration-by-parts has been used along with the identities:

$$\begin{aligned}
\delta J^\alpha =& \frac{\partial J^\alpha}{\partial F_{ij}^\alpha} \delta F_{ij}^\alpha, \\
=& \frac{\partial J^\alpha}{\partial F_{ij}^\alpha} \frac{\partial \delta x_i^\alpha}{\partial X_j}, \\
=& J^\alpha (F_{ji}^\alpha)^{-1} \frac{\partial \delta x_i^\alpha}{\partial X_j},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\delta \left(\frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha} \right) =& \delta \left((\phi_0^\alpha \bar{\rho}_0^\alpha)(\phi^\alpha \bar{\rho}^\alpha)^{-1} \right), \\
=& (\phi_0^\alpha \bar{\rho}_0^\alpha) \delta (\phi^\alpha \bar{\rho}^\alpha)^{-1}, \\
=& - J^\alpha (\phi^\alpha \bar{\rho}^\alpha)^{-1} (\delta \phi^\alpha \bar{\rho}^\alpha + \phi^\alpha \delta \bar{\rho}^\alpha), \\
=& - J^\alpha \left(\frac{\delta \phi^\alpha}{\phi^\alpha} + \frac{\delta \bar{\rho}^\alpha}{\bar{\rho}^\alpha} \right).
\end{aligned} \tag{3.10}$$

Equation (3.8) finishes the variation of the first constraint. As for the volume fraction constraint (3.7) , recognize that this constraint is defined on every material point in the current configuration, and the ϕ^α cannot be completely independent

fields as they will evolve as a function of the current configuration of the mixture. Since we are considering finite deformation and trying to express the variations as functions in the reference configuration, it is non-trivial to compute the variation of the volume fraction constraint. To illustrate this, we first work on the variation of the following equation:

$$\sum_{\alpha} \phi^{\alpha}(\mathbf{x}^{\alpha}) = 1. \tag{3.11}$$

Suppose that there are a small variation $\epsilon\delta\phi^{\alpha}$ on the volume fraction ϕ^{α} and also a small variation $\epsilon\delta\mathbf{x}^{\alpha}$ on the current position $\mathbf{x}^{\alpha}$. Then define

$$\phi^{*\alpha} = \phi^{\alpha} + \epsilon\delta\phi^{\alpha},$$

and

$$\begin{aligned} \mathbf{x}^{*\alpha} =& \chi^{\alpha}(\mathbf{X}^{\alpha}) + \epsilon\delta\mathbf{x}^{\alpha}, \\ =& \boldsymbol{\Upsilon}(\mathbf{X}^{\alpha}, \epsilon), \end{aligned} \tag{3.12}$$

Also define the inverse:

$$\mathbf{X}^{\alpha} = \boldsymbol{\Upsilon}^{-1}(\mathbf{x}^{*\alpha}, \epsilon).$$

With these definitions, the first variation of (3.11) can be written as regular derivative holding $\mathbf{x}^{*\alpha}$ fixed.

$$\begin{aligned} 0 =& \sum_{\alpha} \frac{\mathrm{d}}{\mathrm{d}\epsilon}[\phi^{\alpha}(\mathbf{X}) + \epsilon\delta\phi^{\alpha}]|_{\epsilon=0}, \\ =& \sum_{\alpha} \frac{\mathrm{d}}{\mathrm{d}\epsilon}[\phi^{\alpha}(\boldsymbol{\Upsilon}^{-1}(\mathbf{x}^{*\alpha,\epsilon})) + \epsilon\delta\phi^{\alpha}]|_{\epsilon=0}, \\ =& \sum_{\alpha} \left(\left[\frac{\partial\phi^{\alpha}}{\partial\boldsymbol{\Upsilon}^{-1}} \frac{\partial\boldsymbol{\Upsilon}^{-1}}{\partial\epsilon} \right] \Bigg|_{\epsilon=0} + \delta\phi^{\alpha} \right). \end{aligned} \tag{3.13}$$

In order to evaluate the term $\frac{\partial\boldsymbol{\Upsilon}^{-1}}{\partial\epsilon}$, compute the differential holding $\mathbf{X}^{\alpha}$ fixed on both sides of equation (3.12) :

$$\mathbf{0} = \frac{\partial\boldsymbol{\Upsilon}^{-1}}{\partial\mathbf{x}^{*\alpha}} \mathrm{d}\mathbf{x}^{*\alpha} + \frac{\partial\boldsymbol{\Upsilon}^{-1}}{\partial\epsilon} \mathrm{d}\epsilon,$$

and solve

$$\begin{aligned}\frac{\partial \boldsymbol{\Upsilon}^{-1}}{\partial \epsilon} &= -\left.\frac{\partial \boldsymbol{\Upsilon}^{-1}}{\partial \mathbf{x}^{*\alpha}}\frac{\partial \mathbf{x}^{*\alpha}}{\partial \epsilon}\right|_{\mathbf{X}^\alpha}, \\ &= \frac{\partial \boldsymbol{\Upsilon}^{-1}}{\partial \mathbf{x}^{*\alpha}}\delta\mathbf{x}^\alpha. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13) gives

$$\begin{aligned}
&= \sum_\alpha \left(\left[-\frac{\partial \phi^\alpha}{\partial \boldsymbol{\Upsilon}^{-1}}\frac{\partial \Upsilon^{-1}}{\partial \mathbf{x}^{*\alpha}}\delta\mathbf{x}^\alpha \right] \Big|_{\epsilon=0} + \delta\phi^\alpha \right), \\
&= \sum_\alpha \left(\left[-\frac{\partial \phi^\alpha}{\partial \mathbf{x}^{*\alpha}}\delta\mathbf{x}^\alpha \right] \Big|_{\epsilon=0} + \delta\phi^\alpha \right), \\
&= \sum_\alpha \left(-\frac{\partial \phi^\alpha}{\partial \mathbf{x}^\alpha}\delta\mathbf{x}^\alpha + \delta\phi^\alpha \right), \\
&= \sum_\alpha \left(-(\mathbf{F}^\alpha)^{-\intercal}\frac{\partial \phi^\alpha}{\partial \mathbf{X}}\delta\mathbf{x}^\alpha + \delta\phi^\alpha \right), \\
&= \sum_\alpha \left((\mathbf{F}^\alpha)^{-\intercal}\nabla\phi^\alpha \cdot \delta\mathbf{x}^\alpha - \delta\phi^\alpha \right),
\end{aligned} \tag{3.15}$$

Substituting this result (3.15) into the volume fraction constraint (3.7) gives us the result for variation of the second constraint:

$$\begin{aligned}
\delta C_2 &= \sum_{\alpha=s,f}\int_{\mathcal{B}} p\left((\mathbf{F}^\alpha)^{-\intercal}\nabla\phi^\alpha \cdot \delta\mathbf{x}^\alpha - \delta\phi^\alpha \right) \mathrm{d}V, \\
&= \sum_{\alpha=s,f}\int_{\mathcal{B}_0} pJ^\alpha\left((\mathbf{F}^\alpha)^{-\intercal}\nabla\phi^\alpha \cdot \delta\mathbf{x}^\alpha - \delta\phi^\alpha \right) \mathrm{d}V_0.
\end{aligned} \tag{3.16}$$

Next, we evaluate the variation of the rest of terms in (3.5) . The total kinetic energy of the mixture can be written as

$$T = \frac{1}{2}\int_{\mathcal{B}_0} \rho_0^s \mathbf{v}^s \cdot \mathbf{v}^s + \rho_0^f \mathbf{v}^f \cdot \mathbf{v}^f \, \mathrm{d}V_0,$$

where $\mathbf{v}^s$ and $\mathbf{v}^f$ are the velocities of solid and fluid respectively:

$$\mathbf{v}^\alpha = \frac{\partial \chi^\alpha(\mathbf{X}, t)}{\partial t}, \qquad \alpha = s, f$$

Taking the first variation of the kinetic energy during the application of Hamilton's principle and integrating in time give us

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T \mathrm{d}t &= \int_{t_1}^{t_2} \frac{\partial T}{\partial \mathbf{v}^s} \cdot \delta \mathbf{v}^s + \frac{\partial T}{\partial \mathbf{v}^f} \cdot \mathbf{v}^f \mathrm{d}t, \\
&= \left[\frac{\partial T}{\partial \mathbf{v}^s} \cdot \delta \mathbf{x}^s + \frac{\partial T}{\partial \mathbf{v}^f} \cdot \delta \mathbf{x}^f \right] \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \mathbf{v}^s} \right) \cdot \delta \mathbf{x}^s + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \mathbf{v}^f} \right) \cdot \delta \mathbf{x}^f \mathrm{d}t, \\
&= \int_{t_1}^{t_2} \int_{\mathcal{B}_0} \rho_0^s \mathbf{a}^s \cdot \delta \mathbf{x}^s + \rho_0^f \mathbf{a}^f \cdot \delta \mathbf{x}^f \, \mathrm{d}V_0 \, \mathrm{d}t, \qquad (3.17)
\end{aligned}$$

where integration-by-parts has been used and the first term in the second of (3.17) vanishes due to the standard assumption in the application of Hamilton's principle that the variations are fixed at the endpoints of the prescribed time interval $[t_1, t_2]$ and

$$\mathbf{a}^\alpha = \frac{\partial^2 \chi^\alpha(\mathbf{X}, t)}{\partial^2 t}, \qquad \alpha = s, f.$$

The total potential energy of the binary mixture is given by

$$U = \int_{\mathcal{B}_0} \rho_0^s e^s(\bar{\rho}^s, \mathbf{F}^s) + \rho^f e^f(\bar{\rho}^f) \, \mathrm{d}V_0, \qquad (3.18)$$

where the e^s and e^f are the internal energies per unit mass of solid and fluid respectively. Notice that the internal energy of the solid is treated as a function of deformation gradient and solid local density. This is because the density of the solid can change independently of the deformation due to the internal presence of the pores containing the fluid whose volume can change regardless of solid deformation.

Taking the first variation of (3.18) gives

$$\begin{aligned}
\delta U &= \int_{\mathcal{B}_0} \rho_0^s \left(\frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^s + \frac{\partial e^s}{\partial F_{ij}^s} \frac{\partial \delta x_i^s}{\partial X_j} \right) + \rho_0^f \frac{\partial e^f}{\partial \bar{\rho}^f} \delta \bar{\rho}^f \, \mathrm{d}V_0, \\
&= \int_{\mathcal{B}_0} \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^s + P_{ij}^{\prime s} \frac{\partial \delta x_i^s}{\partial X_j} + \rho_0^f \frac{\partial e^f}{\partial \bar{\rho}^f} \delta \bar{\rho}^f \, \mathrm{d}V_0, \\
&= \int_{\mathcal{B}_0} \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^s + \rho_0^f \frac{\partial e^f}{\partial \bar{\rho}^f} \delta \bar{\rho}^f - \frac{\partial P_{ij}^{\prime s}}{\partial X_j} \delta x_i^s \, \mathrm{d}V_0 + \int_{\partial B_0} P_{ij}^{\prime s} N_j \delta x_i^s \mathrm{d}S_0, \\
&= \int_{\mathcal{B}_0} \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^s + \rho_0^f \frac{\partial e^f}{\partial \bar{\rho}^f} \delta \bar{\rho}^f - (\nabla \cdot \mathbf{P}^{\prime s}) \cdot \delta \mathbf{x}^s \, \mathrm{d}V_0 + \int_{\partial B_0} \mathbf{P}^{\prime s} \mathbf{N} \delta \mathbf{x}^s \mathrm{d}S_0, \quad (3.19)
\end{aligned}$$

where integration-by-parts has been used along with the notation:

$$P_{ij}^{\prime s} = \rho_0^s \left. \frac{\partial e^s}{\partial F_{ij}^s} \right|_{\bar{\rho}^s}, \quad (3.20)$$

which is very similar to the definition of the first Piola-Kirchhoff stress and will be further discussed later. The first variation of virtual work for this binary mixture can be written as

$$\begin{aligned}
\delta W = &\int_{\mathcal{B}_0} (\rho_0^s \mathbf{G} + \mathbf{H}^s) \cdot \delta \mathbf{x}^s + \rho_0^f \mathbf{G} + \mathbf{H}^f) \cdot \delta \mathbf{x}^f \, \mathrm{d}V_0 \\
&- \int_{\partial \mathcal{B}_0} p_0^f J^f (\mathbf{F}^f)^{-\intercal} \mathbf{N} \cdot \delta \mathbf{x}^f + \mathbf{S}_0^s \cdot \delta \mathbf{x}^s \mathrm{d}S_0, \quad (3.21)
\end{aligned}$$

where $\mathbf{G}$ is a body force density per unit reference volume that is assumed to be identical for both the solid and fluid, e.g. gravity. $\mathbf{H}^s$ and $\mathbf{H}^f$ are drag terms that identify the interactive body forces per unit reference volume that the solid and fluid constituents impose one each other. These forces internally balance:

$$\mathbf{H}^s + \mathbf{H}^f = \mathbf{0}.$$

p_0^f is a scalar force per unit current area that acts normal to the surface $\partial \mathcal{B}$ on the fluid, i.e. pressure, as shown it has been pulled back through $J^f (\mathbf{F}^f)^{-\intercal}$ to the reference configuration. $\mathbf{S}_0^s$ is a traction per unit reference volume that is applied to the surface $\partial \mathcal{B}_0$ of the solid.

Finally, substitue all the results we have for these varations (3.8) , (3.15) ,

(3.17) , (3.19) , (3.21) , into extended Hamilton's principle (3.5) , and evoke the arbitrariness of $\delta\mathbf{x}^\alpha$, $\delta\phi^\alpha$, and $\delta\bar{\rho}^\alpha$. We have the following equations describing the mixture's motion:

$$\rho_0^s\mathbf{a}^s = \rho_0^s\mathbf{G} + \mathbf{H}^s - \nabla\cdot\left(\lambda^s J^s(\mathbf{F}^s)^{-\intercal}\right) + pJ^s(\mathbf{F}^s)^{-\intercal}\nabla\phi^s + \nabla\cdot\mathbf{P}'^s, \tag{3.22a}$$

$$\rho_0^f\mathbf{a}^f = \rho_0^f\mathbf{G} + \mathbf{H}^f - \nabla\cdot\left(\lambda^f J^f(\mathbf{F}^f)^{-\intercal}\right) + pJ^f(\mathbf{F}^f)^{-\intercal}\nabla\phi^f, \tag{3.22b}$$

$$\lambda^s = \phi^s(\bar{\rho}^s)^2\frac{\partial e^s}{\partial\bar{\rho}^s}, \tag{3.22c}$$

$$\lambda^f = \phi^f(\bar{\rho}^f)^2\frac{\partial e^f}{\partial\bar{\rho}^f}, \tag{3.22d}$$

$$\lambda^s = p\phi^s, \tag{3.22e}$$

$$\lambda^f = p\phi^f, \tag{3.22f}$$

with the boundary conditions:

$$p_0^f = \lambda^f, \tag{3.23a}$$

$$\mathbf{S}_0^s = \mathbf{P}'^s\mathbf{N}. \tag{3.23b}$$

Eliminating λ^s and λ^f from (3.22) and (3.23) , we have

$$\rho_0^s\mathbf{a}^s = \rho_0^s\mathbf{G} + \mathbf{H}^s - \nabla\cdot\left(\phi^s pJ^s(\mathbf{F}^s)^{-\intercal}\right) + pJ^s(\mathbf{F}^s)^{-\intercal}\nabla\phi^s + \nabla\cdot\mathbf{P}'^s, \tag{3.24a}$$

$$\rho_0^f\mathbf{a}^f = \rho_0^f\mathbf{G} + \mathbf{H}^f - \nabla\cdot\left(\phi^f pJ^f(\mathbf{F}^f)^{-\intercal}\right) + pJ^f(\mathbf{F}^f)^{-\intercal}\nabla\phi^f, \tag{3.24b}$$

$$p = (\bar{\rho}^s)^2\frac{\partial e^s}{\partial\bar{\rho}^s} = (\bar{\rho}^f)^2\frac{\partial e^f}{\partial\bar{\rho}^f}, \tag{3.24c}$$

with

$$p_0^f = p\phi^f, \tag{3.25a}$$

$$\mathbf{S}_0^s = \mathbf{P}'^s\mathbf{N}. \tag{3.25b}$$

And these can be further simplified to

$$\rho_0^s \mathbf{a}^s = \rho_0^s \mathbf{G} + \mathbf{H}^s - \phi^s \nabla \cdot \left(p J^s (\mathbf{F}^s)^{-\intercal} \right) + \nabla \cdot \mathbf{P}'^s, \tag{3.26a}$$

$$\rho_0^f \mathbf{a}^f = \rho_0^f \mathbf{G} + \mathbf{H}^f - \phi^f \nabla \cdot \left(p J^f (\mathbf{F}^f)^{-\intercal} \right), \tag{3.26b}$$

$$p = (\bar{\rho}^s)^2 \frac{\partial e^s}{\partial \bar{\rho}^s} = (\bar{\rho}^f)^2 \frac{\partial e^f}{\partial \bar{\rho}^f}, \tag{3.26c}$$

with

$$p_0^f = p \phi^f, \tag{3.27a}$$

$$\mathbf{S}_0^s = \mathbf{P}'^s \mathbf{N}. \tag{3.27b}$$

As a remark, the pressure of the fluid is not involved during the entire derivation, because we only use the local density of fluid to describe the fluid's motion. However, equation (3.26c) shows that p, which is introduced as a Lagrange multiplier in the extended Hamilton's principle, turns out to be the fluid's pressure $(\bar{\rho}^f)^2 \frac{\partial e^f}{\partial \bar{\rho}^f}$ according to the common thermodynamic definition. This result makes sense because the pressure in the binary mixture is the reaction force describing the interaction between fluid and solid, which is based on volume fraction constraint. This method provides a natural way of involving fluid pressure in the momentum balance equations for porous media without presupposing an "effective stress."

Equations (3.26) and (3.27) along with appropriate constitutive models for the internal energies and the drag terms as well as statements of conservation of mass can be used to solve for the current position $\mathbf{x}^s$, $\mathbf{x}^f$ and local densities $\bar{\rho}^s$, $\bar{\rho}^f$, therefore establishing a complete theory for this porous media as a binary mixture.

In considerations of geomaterials, it is popular to assume that the fluid follows the same trajectory as the solid, i.e. $\mathbf{F}^s = \mathbf{F}^f$ as the solid allowing for relative motion between the two constituents along the path. Then adding (3.26a) and (3.26b) together under this assumption gives

$$\begin{aligned} \rho_0^s \mathbf{a}^s + \rho_0^f \mathbf{a}^f &= \rho_0 \mathbf{G} - \nabla \cdot \left(p J^s (\mathbf{F}^s)^{-\intercal} \right) + \nabla \cdot \mathbf{P}'^s, \\ &= \rho_0 \mathbf{G} + \nabla \cdot \left(\mathbf{P}'^s - p J^s (\mathbf{F}^s)^{-\intercal} \right), \end{aligned} \tag{3.28}$$

where

$$\rho_0 = \rho_0^s + \rho_0^f.$$

Comparing equation (3.28) with the general finite deformation momentum balance equation, we notice that the second term in the right side of (3.28) plays the role of the first Piola-Kirchhoff stresses as in the general momentum balance equation. Further discussion about the effective stress and meaning of the term $\mathbf{P}'^s$ will made be made in the following section.

3.3 Correspondence to Biot's Theory

Given the constitutive equation for the total stress (2.2) in Biot's theory along with the definition of Biot coefficient (2.3) , it is straightforward to write out the momentum balance equation of Biot's theory:

$$\rho^s \mathbf{a}^s + \rho^f \mathbf{a}^f = \rho \mathbf{G} + \nabla \cdot (\mathbb{C} : \epsilon - np), \tag{3.29}$$

where ρ^s, ρ^f and ρ are solid density, fluid density and total density respectively; $\mathbf{G}$ is the body force density; $\mathbb{C}$ is the stiffness tensor for solid skeleton; ϵ is the strain tensor for porous media; n is Biot's coefficient; p is pore pressure; and Biot define his effective stress σ' as

$$\sigma' = \mathbb{C} : \epsilon, \tag{3.30}$$

Comparing (3.29) with our results (3.28) derived from extended Hamilton's principle, we notice that every term in (3.28) can be easily seen to be approximately equal to the corresponding term in (3.29) if the deformation is assumed to be small, except for the last term. So, to prove the correspondence of our result from extended Hamilton's principle to Biot's theory, we only need to show following equation is satisfied considering small deformation:

$$\mathbf{P}'^s - pJ^s(\mathbf{F}^s)^{-\intercal} = \mathbb{C} : \epsilon - np. \tag{3.31}$$

First, we introduce our definition of effective stress based on the derivation from Hamilton's principle. During the derivation in the previous section, we wrote our solid energy density e^s as a function of the deformation gradient $\mathbf{F}^s$ and local

density $\bar{\rho}^s$:

$$e^s = e^s(\mathbf{F}^s, \bar{\rho}^s).$$

The inherent reason why the internal energy of the solid cannot only be described by displacement is that the displacement vector $\mathbf{x}^s$ in mixture theory is the macroscopic displacement of the mixture, it does not contain any information about the micro-deformation at each point which can be described by the local density at each point. Then we can write the total differential of solid energy per unit mass $\rho_0^s e^s$ as

$$\begin{aligned}\rho_0^s \mathrm{d}e^s =& \rho_0^s \frac{\partial e^s}{\partial F_{ij}} \mathrm{d}F_{ij} + \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \mathrm{d}\bar{\rho}^s, \\ =& P'^s_{ij} \mathrm{d}F_{ij} + \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \mathrm{d}\bar{\rho}^s. \end{aligned} \tag{3.32}$$

Therefore, there are virtual forces describing the change of solid internal energy. One describes the rate of the change of energy when changing the displacement field and the other describes the rate of the change of energy when changing the local density. Since the first virtual force describes the relation between energy and displacement, it has the same feature as stress. So we call this virtual force the effective stress. The effective stress is defined as the partial derivative of the solid internal energy with respect to the deformation gradient. However, the partial derivative is dependent on what is chosen to be the second variable of the solid internal energy function. In the case of (3.32) , the second variable is chosen to be the density, so we call P'^s_{ij} the effective stress holding local density fixed. In this way of defining effective stress, there are many effective stresses depending on what variable you choose as the second independent variable.

The effective stress defined by Biot [1962] and Coussy [1995] is somehow defined on the stress level. They first assume that the energy of the binary mixture has the general relation with the total stress and strain:

$$\mathrm{d}e(\mathbf{x}^s, p) = \sigma(\mathbf{x}^s, p) \cdot \mathrm{d}\epsilon, \tag{3.33}$$

where $e(\mathbf{x}^s, p)$ is the total energy of the porous media, $\sigma(\mathbf{x}^s, p)$ is the total stress.

Then the two variable can be approximately linearly separated as

$$\begin{aligned} \mathrm{d}\sigma(\mathbf{x}^s, p) =& \mathrm{d}\sigma' - n\mathrm{d}p, \\ =& \mathbb{C} : \mathrm{d}\epsilon - n\mathrm{d}p, \end{aligned} \tag{3.34}$$

where n is Biot coefficient and σ' is the effective stress defined by Biot and Coussy. And if it is further assumed that the solid is isotropic linear thermoporoelasticity [Coussy, 2004]

$$\sigma(\mathbf{x}^s, p) = \mathbb{C} : \epsilon - np.$$

However, the assumption of (3.33) is not obviously justified. Here we provide a way of verifying (3.33) rigorously. From the derivation in the previous section, we have

$$\rho_0^s \mathrm{d}e^s = \rho_0^s \frac{\partial e^s}{\partial \bar{\rho}^s} \mathrm{d}\bar{\rho}^s + \rho_0^s \frac{\partial e^s}{\partial F_{ij}} \mathrm{d}F_{ij}. \tag{3.35}$$

The constraint of volume fraction gives us the relation to Lagrange multiplier p (pressure):

$$p = (\bar{\rho}^s)^2 \frac{\partial e^s}{\partial \bar{\rho}^s}.$$

Substituting into the (3.35) gives

$$\begin{aligned} \rho_0^s \mathrm{d}e^s =& \frac{\rho_0^s p}{(\bar{\rho}^s)^2} \mathrm{d}\bar{\rho}^s + P'_{ij} \mathrm{d}F_{ij}, \\ =& \frac{\rho_0^s p}{(\bar{\rho}^s)^2} \mathrm{d}(\frac{1}{v^s}) + P'_{ij} \mathrm{d}F_{ij}, \\ =& -\rho_0^s \, p \, \mathrm{d}v^s + P'_{ij} \mathrm{d}F_{ij}, \end{aligned} \tag{3.36}$$

where v^s is the solid volume per unit mass. And similarly for fluid, we also have

$$\rho_0^f \mathrm{d}e^f = -\rho_0^f p \mathrm{d}v^f. \tag{3.37}$$

Adding (3.36) and (3.37) together gives

$$\begin{aligned}\rho_0^s \mathrm{d}e^s + \rho_0^f \mathrm{d}e^f =& -\rho_0^s\, p\, \mathrm{d}v^s + P'_{ij}\mathrm{d}F^s_{ij} - \rho_0^f\, p\, \mathrm{d}v^f,\\ \mathrm{d}e =& P'_{ij}\mathrm{d}F^s_{ij} - \rho_0^s\, p\mathrm{d}\left(\frac{J^s\phi^s}{\rho_0^s}\right) - \rho_0^f\, p\mathrm{d}\left(\frac{J^f\phi^f}{\rho_0^f}\right),\\ =& P'_{ij}\mathrm{d}F^s_{ij} - p\phi^s\mathrm{d}J^s - p\phi^f\mathrm{d}J^f - pJ^s\mathrm{d}\phi^s - pJ^f\mathrm{d}\phi^f,\\ =& \left(P'_{ij} - p\phi^s J^s(F^s_{ji})^{-1} - p\phi^f J^s(F^s_{ji})^{-1}\right)\mathrm{d}F^s_{ij} - pJ(\mathrm{d}\phi^s + \mathrm{d}\phi^f),\end{aligned} \tag{3.38}$$

where the mass conservation (3.4) has been used and the assumption

$$\mathbf{F}^s = \mathbf{F}^f,$$

has also been used. Notice that equation (3.38) is in the reference configuration, where $\phi^s(\mathbf{X},t) + \phi^f(\mathbf{X},t)$ is not necessarily be equal to 1. Only when deformation is small enough, we can say that the sum of volume fractions in the reference configuration is approximately equal to 1:

$$\begin{aligned}\phi^s(\mathbf{X},t) + \phi^f(\mathbf{X},t) \approx& 1,\\ \delta\phi^s + \delta\phi^f =& 0.\end{aligned}$$

Substituting into (3.38) gives

$$\mathrm{d}e = (P'_{ij} - pJ^s(F^s_{ji})^{-1})\mathrm{d}F^s_{ij},$$

which validates the assumption (3.33) .

Returning to the correspondence of our result to Biot's theory, i.e. equation (3.31) . According to equation (3.36) , the differential of the energy density of solid per unit mass $\rho_0^s \mathrm{d}e^s$ can be written as

$$\rho_0^s \mathrm{d}e^s = -\rho_0^s\, p\, \mathrm{d}v^s + P'_{ij}\mathrm{d}F_{ij}. \tag{3.39}$$

Here the superscript s on the deformation gradient F_{ij} is neglected because of the assumption $\mathbf{F}^s = \mathbf{F}^f$. The volume of solid per unit mass v^s can be written as a

function of pressure p and the deformation gradient $\mathbf{F}$, so the differential of v^s is

$$\mathrm{d}v^s = \left.\frac{\partial v^s}{\partial p}\right|_{\mathbf{F}} \mathrm{d}p + \left.\frac{\partial v^s}{\partial F_{ij}}\right|_{p} \mathrm{d}F_{ij}.$$

Substituting into (3.39) gives

$$\begin{aligned}\rho_0^s \mathrm{d}e^s =& -\rho_0^s\, p \left(\left.\frac{\partial V^s}{\partial p}\right|_{\mathbf{F}} \mathrm{d}p + \left.\frac{\partial V^s}{\partial F_{ij}}\right|_{p} \mathrm{d}F_{ij} \right) + P'_{ij}\mathrm{d}F_{ij}, \\ =& -\rho_0^s\, p \left.\frac{\partial v^s}{\partial p}\right|_{\mathbf{F}} \mathrm{d}p + \left(P'_{ij} - p\,\frac{1}{v_0^s}\left.\frac{\partial v^s}{\partial F_{ij}}\right|_{p} \right) \mathrm{d}F_{ij}, \end{aligned} \tag{3.40}$$

where

$$v_0^s := \frac{1}{\rho_0^s}.$$

Hence, if we define $\mathbf{P}''$ as the effective stress holding pore pressure p fixed, then the effective stress P''_{ij} can be written as

$$P''_{ij} = \rho_0^s \left.\frac{\partial e^s}{\partial F_{ij}}\right|_{p} = P'_{ij} - p\,\frac{1}{v_0^s}\left.\frac{\partial v^s}{\partial F_{ij}}\right|_{p}.$$

And the term $\left.\frac{1}{v_0^s}\frac{\partial v^s}{\partial F_{ij}}\right|_{p}$ describes the volume change of the real mineral solid versus the volume change of the entire skeleton when holding pressure fixed. In the case that the deformation is small and isotropic, we can approximate this term by further assuming that the solid material v^s is only dependent on the Jacobian determinant $J = \det(\mathbf{F})$ without any dependency on other information contained in deformation gradient $\mathbf{F}$ e.g. the distortion. With this assumption, we further rewrite the term $\left.\frac{1}{v_0^s}\frac{\partial v^s}{\partial F_{ij}}\right|_{p}$ as

$$\begin{aligned}\left.\frac{1}{v_0}\frac{\partial v^s}{\partial F_{ij}}\right|_{p} \approx& \left.\frac{1}{v_0}\frac{\partial v^s}{\partial J}\right|_{p} \frac{\partial J}{\partial F_{ij}}, \\ =& \left.\frac{1}{v_0}\frac{\partial v^s}{\partial J}\right|_{p} J(F_{ji})^{-1}, \\ \approx& \frac{K^*}{K^s} J(F_{ji})^{-1}, \end{aligned} \tag{3.41}$$

where K^* is the bulk modulus of the solid skeleton and K^s is the real mineral solid bulk modulus. Substituting (3.41) into the left side of (3.31) gives

$$\begin{aligned}\mathbf{P}' - pJ^s(\mathbf{F}^s)^{-\intercal} =& \mathbf{P}'' - p\left(1 - \frac{K^*}{K^s}\right) J^s(\mathbf{F}^s)^{-\intercal}, \\ =& \mathbf{P}'' - npJ^s(\mathbf{F}^s)^{-\intercal}. \end{aligned} \tag{3.42}$$

which is exactly same as the form of the right side of (3.31) after pulled back to the reference configuration. So, the correspondence of our results to Biot's theory is verified which also presents new and rigorous insights into effective stress and Biot's coefficient, and validates the variational method based on extended Hamilton's principle as a method for deriving equations of motion for the binary mixture.

Chapter 4

Poroelasticity Peridynamic Equations Derived using Hamilton's Principle

The previous chapter presented a finite deformation theory for poroelasticity whose correspondence to classical Biot's theory is also validated. However, since it is performed in the framework of continuum mechanics, it has two important assumptions: the displacement and the geometry are continuous; and the deformation is smooth and homogeneous. In this chapter, we remove these assumptions and develop a nonlocal theory for finite deformation poroelasticity based on state-based peridynamic theory that is introduced in Section 2.2.2. We also start by applying extended Hamilton principle.

With the state-based peridynamic concepts that are presented in Chapter 2.2.2, we use $\mathbf{X}$ to denote a material point in the reference configuration of the binary mixture body $\mathcal{B}_0$ and $\mathbf{Q}$ denote another material point inside the family $\mathcal{H}(\mathbf{X})$ in the reference configuration. Since there are both solid and fluid constituents in every material point in the reference configuration, let $\mathbf{x}^s$ denote the position vector of the solid constituent of material point $\mathbf{X}$ in the current configuration after

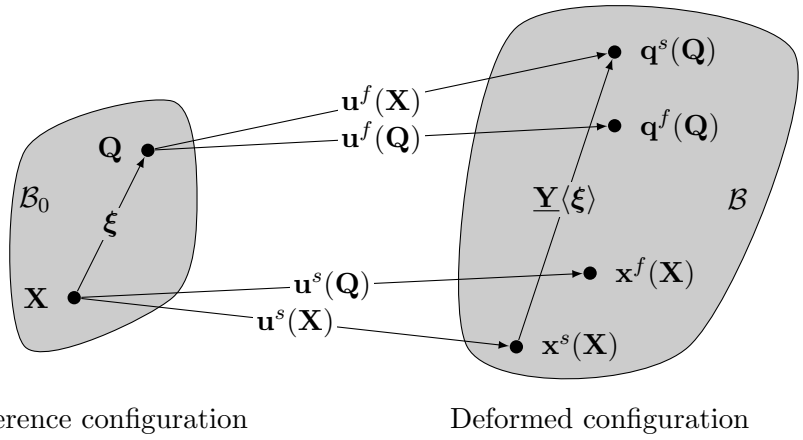

Figure 4.1: Schematic description of peridynamic kinematics in binary mixture

deformations and do the same notations for material point $\mathbf{Q}$. We can write

$$\begin{aligned} \mathbf{x}^s =& \mathbf{x}^s(\mathbf{X}), \qquad \mathbf{q}^s = \mathbf{q}^s(\mathbf{Q}), \\ \mathbf{x}^f =& \mathbf{x}^f(\mathbf{X}), \qquad \mathbf{q}^f = \mathbf{q}^f(\mathbf{Q}). \end{aligned}$$

Let $\mathbf{u}^s$ and $\mathbf{u}^f$ denote the displacement vector for solid and fluid respectively:

$$\begin{aligned} \mathbf{u}^s(\mathbf{X}) = \mathbf{x}^s - \mathbf{X}, \qquad \mathbf{u}^s(\mathbf{Q}) = \mathbf{q}^s - \mathbf{Q}, \\ \mathbf{u}^f(\mathbf{X}) = \mathbf{x}^f - \mathbf{X}, \qquad \mathbf{u}^f(\mathbf{Q}) = \mathbf{q}^f - \mathbf{Q}, \end{aligned}$$

and define the bond $\boldsymbol{\xi}$ and deformation vector-state $\underline{\mathbf{Y}}$ of the solid and fluid as

$$\begin{aligned} \boldsymbol{\xi} =& \mathbf{Q} - \mathbf{X}, \\ \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle =& \mathbf{q}^s(\mathbf{Q}) - \mathbf{x}^s(\mathbf{X}). \end{aligned}$$

Note that we only denote the deformation vector-state for the solid constituent, because the motion of fluid is independent of fluid's deformation. An schematic description of peridynamic kinematics in binary mixture is shown is Figure respectively. A schematic description of a peridynamic body and the family at $\mathbf{X}$ is shown in Figure 4.1.

Using the same notations as in the previous chapter, the extended Hamilton's

principle states that among admissible motions, the actual body is such that

$$\int_{t_1}^{t_2} \delta(T - V) + \delta W + \sum_k \delta C_k \mathrm{d}t = 0, \tag{4.1}$$

where the extra constraints C_1 and C_2 are

$$C_1 = \sum_{\alpha=s,f} \int_{\mathcal{B}_0} \lambda^\alpha \left(J^\alpha - \frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha} \right), \tag{4.2}$$

$$C_2 = \int_{\mathcal{B}} p \left(\sum_{\alpha=s,f} \phi^\alpha - 1 \right) \mathrm{d}V. \tag{4.3}$$

An important difference between the derivation in the previous chapter and the following derivation based on peridynamics is that do not assume spatial gradient exist when evaluating the first variations in (4.1) . Using the results (3.17) in the previous chapter, the first variation of the kinetic energy δT can be written as

$$\delta T \mathrm{d}t = \int_{\mathcal{B}_0} \rho_0^s \mathbf{a}^s \cdot \delta \mathbf{x}^s + \rho_0^f \mathbf{a}^f \cdot \delta \mathbf{x}^f \, \mathrm{d}V_0. \tag{4.4}$$

Also, neglecting the boundary conditions in (3.21) , the first variation of virtual work δW can be written as

$$\delta W = \int_{\mathcal{B}_0} (\rho_0^s \mathbf{G} + \mathbf{H}^s) \cdot \delta \mathbf{x}^s + (\rho_0^f \mathbf{G} + \mathbf{H}^f) \cdot \delta \mathbf{x}^f \, \mathrm{d}V_0. \tag{4.5}$$

As for the internal energy of the binary mixture V, it can be written as the sum of solid energy V^s and the fluid energy V^f:

$$V = V^s + V^f, \tag{4.6}$$

and we first focus on the internal energy of solid constituent. Let V^s denote the total energy of the solid constituent, defined in the reference configuration, in the bounded region $\mathcal{B}_0$. Let $\Psi^s(\mathbf{X}, t)$ denote the internal energy density defined in the reference configuration (energy per unit reference volume), and it can also be written as a function of the deformation at $\mathbf{X}$ which can be described nonlocally by vector-state

$\underline{\mathbf{Y}}(\mathbf{X})$ and solid local density $\bar{\rho}^s$:

$$V^s = \int_{\mathcal{B}_0} \Psi^s(\underline{\mathbf{Y}}(\mathbf{X}), \bar{\rho}^s)\, \mathrm{d}\mathbf{X},$$

where the notation $\mathrm{d}\mathbf{X}$ means that the dummy variable of the integration is vector $\mathbf{X}$ for clarity, but the value of $\mathrm{d}\mathbf{X}$ is still an infinitesimal volume. Taking the first variation of V^s gives

$$\begin{aligned}
\delta V^s &= \int_{B_0} \delta\Psi^s(\underline{\mathbf{Y}}^s(\mathbf{X}), \overline{\rho}^s)\, \mathrm{d}\mathbf{X}, \\
&= \int_{B_0}\int_{B_0} \underline{\nabla}\Psi^s(\mathbf{X}, \overline{\rho}^s)\langle\boldsymbol{\xi}\rangle \cdot \delta\underline{\mathbf{Y}}^s \, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\mathbf{X} + \int_{B_0} \frac{\partial\Psi^s(\mathbf{X}, \overline{\rho}^s)}{\partial\overline{\rho}^s}\delta\overline{\rho}^s \, \mathrm{d}\mathbf{X}, \\
&= \int_{B_0}\int_{B_0} \underline{\nabla}\Psi^s(\mathbf{X}, \overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle \cdot (\delta\mathbf{u}^s(\mathbf{Q}) - \delta\mathbf{u}^s(\mathbf{X}))\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\mathbf{X} \\
&\quad + \int_{B_0} \frac{\partial\Psi^s(\mathbf{X}, \overline{\rho}^s)}{\partial\overline{\rho}^s}\delta\overline{\rho}^s \, d\mathbf{X}, \\
&= \int_{B_0} \left(\int_{B_0} \underline{\nabla}\Psi^s(\mathbf{Q}, \overline{\rho}^s(\mathbf{Q}))\langle -\boldsymbol{\xi}\rangle - \underline{\nabla}\Psi^s(\mathbf{X}, \overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle \, \mathrm{d}\boldsymbol{\xi} \right) \cdot \delta\mathbf{u}^s(\mathbf{X})\, \mathrm{d}\mathbf{X} \\
&\quad + \int_{B_0} \frac{\partial\Psi^s(\mathbf{X}, \overline{\rho}^s)}{\partial\overline{\rho}^s}\delta\overline{\rho}^s \, \mathrm{d}\mathbf{X},
\end{aligned} \tag{4.7}$$

where the integral over $\mathrm{d}\boldsymbol{\xi}$ in the second of (4.7) arises due to properties of the Fréchet derivative of functions of peridynamic states as shown in (2.10) . Sequential change-of-variable substitutions followed by a change in order-of-integration was used in the manipulations on the first term between the third and forth of (4.7) . Note that the variation of $\delta\mathbf{u}^s$ is identical as $\delta\mathbf{x}^s$ since

$$\delta\mathbf{u}^s = \delta(\mathbf{x}^s - \mathbf{X}) = \delta\mathbf{x}^s.$$

As for internal energy of fluid constituent V^f, we can also express V^f in terms of energy density of fluid $\Psi^f(\bar{\rho}^f)$, which is only dependent on fluid's local density:

$$V^f = \int_{\mathcal{B}_0} \Psi^f(\bar{\rho}^f)\mathrm{d}\mathbf{X}.$$

Since $\Psi^f(\bar{\rho}^f)$ does not have the dependency on deformation, the first variation of

fluid energy δV^f is quite straightforward:

$$\begin{aligned}\delta V^f &= \int_{B_0} \delta\Psi^f(\overline{\rho}^f)\,\mathrm{d}\mathbf{X},\\ &= \int_{B_0} \frac{\partial\Psi^f(\overline{\rho}^f)}{\partial\overline{\rho}^f}\delta\overline{\rho}^f\,\mathrm{d}\mathbf{X}.\end{aligned} \tag{4.8}$$

Following the same steps in (3.8) when evaluating the first variation of the first constraint δC_1, we can write write δC_1 as

$$\delta C_1 = \sum_\alpha \int_{B_0} \lambda^\alpha \delta J + \lambda^\alpha J\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right)\mathrm{d}\mathbf{X}. \tag{4.9}$$

where the assumption that the fluid follows the same trajectory as solid i.e. $J = J^s = J^f$ has been used. Then, we apply the same method that we used to evaluate $\delta\Psi^s$ in (4.7) to evaluating δJ:

$$\begin{aligned}\delta J &= \int_{B_0} \underline{\nabla} J(\underline{\mathbf{Y}}(\boldsymbol{X}))\langle\boldsymbol{\xi}\rangle\cdot\delta\underline{\mathbf{Y}}\,\mathrm{d}\mathbf{X},\\ &= \int_{B_0} \underline{\nabla} J(\underline{\mathbf{Y}}(\boldsymbol{X}))\langle\boldsymbol{\xi}\rangle\cdot(\delta\mathbf{u}^\alpha(\mathbf{Q}) - \delta\mathbf{u}^\alpha(\mathbf{X}))\,\mathrm{d}\mathbf{X}.\end{aligned} \tag{4.10}$$

Substituting into (4.9) gives

$$\begin{aligned}\delta C_1 =& \sum_\alpha \int_{B_0}\int_{B_0} \lambda^\alpha(\mathbf{X})\underline{\nabla} J(\underline{\mathbf{Y}}(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\cdot(\delta\mathbf{u}^\alpha(\mathbf{Q}) - \delta\mathbf{u}^\alpha(\mathbf{X}))\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\mathbf{X}\\ &+ \int_{B_0}\lambda^\alpha J\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right)\mathrm{d}\mathbf{X},\\ =& \sum_\alpha \int_{B_0}\left(\int_{B_0}\lambda^\alpha(\mathbf{Q})\underline{\nabla} J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle - \lambda^\alpha(\mathbf{X})\underline{\nabla} J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\,\mathrm{d}\boldsymbol{\xi}\right)\cdot\delta\mathbf{u}^\alpha(\mathbf{X})\,\mathrm{d}\mathbf{X}\\ &+ \int_{B_0}\lambda^\alpha J\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right)\mathrm{d}\mathbf{X}.\end{aligned} \tag{4.11}$$

As for the volume fraction constraint, we also follow the same steps in (3.15) in the previous chapter, we can write δC_2 as

$$\delta C_2 = \int_{\mathcal{B}_0} p\sum_\alpha J\left(\nabla_{\mathbf{x}}\phi^\alpha\cdot\delta\mathbf{u}^\alpha - \delta\phi^\alpha\right)\mathrm{d}\mathbf{X}, \tag{4.12}$$

where the subscript $\mathbf{x}$ in the term $\nabla_{\mathbf{x}}\phi^{\alpha}$ means that the spatial gradient is in the current configuration, i.e. $\frac{\partial \phi^{\alpha}}{\partial \mathbf{x}}$. Note that in equation (4.12) , the term $\nabla_{\mathbf{x}}\phi^{\alpha}$ is now still a local term involving spatial gradient, but we will illustrate how we evaluate this term nonlocally later.

Substituting (4.4) , (4.5) , (4.6) , (4.7) , (4.8) , (4.11) , and (4.12) into (4.1) , and evoking the arbitrariness of $\delta\mathbf{x}^{\alpha}$, $\delta\phi$, and $\bar{\rho}^{\alpha}$, the motion of the binary mixture is described by

$$\begin{aligned}\rho_0^s\ddot{\mathbf{u}}^s =&\rho_0^s\mathbf{G}+\mathbf{H}^s+\int_{B_0}\left(\lambda^s(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-\lambda^s(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)d\boldsymbol{\xi}+pJ\underline{\nabla}_{\mathbf{x}}\phi^s\\&-\int_{B_0}\underline{\nabla}\Psi^s(\mathbf{Q},\bar{\rho}^s(\mathbf{Q}))\langle-\boldsymbol{\xi}\rangle-\underline{\nabla}\Psi^s(\mathbf{X},\bar{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\,d\boldsymbol{\xi},\end{aligned}\tag{4.13}$$

$$\rho_0^f\ddot{\mathbf{u}}^f =\rho_0^f\mathbf{G}+\mathbf{H}^f+\int_{B_0}\left(\lambda^f(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-\lambda^f(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)d\boldsymbol{\xi}+pJ\underline{\nabla}_{\mathbf{x}}\phi^f,\tag{4.14}$$

$$\lambda^s =\frac{\bar{\rho}^s}{J}\cdot\frac{\partial\Psi^s(\mathbf{X},\bar{\rho}^s)}{\partial\bar{\rho}^s},\tag{4.15}$$

$$\lambda^f =\frac{\bar{\rho}^f}{J}\cdot\frac{\partial\Psi^f(\mathbf{X},\bar{\rho}^f)}{\partial\bar{\rho}^f},\tag{4.16}$$

$$\lambda^s =p\phi^s,\tag{4.17}$$

$$\lambda^f =p\phi^f.\tag{4.18}$$

It is often physically justified due to the attenuation of nonlocal interactions at a distance to specify a characteristic length scale in a peridynamic model δ such that

$$\underline{\nabla}\Psi^{\alpha}\langle\boldsymbol{\xi}\rangle=0,\qquad\forall\,|\boldsymbol{\xi}|>\delta\qquad\alpha=s,f$$

$$\underline{\nabla}J\langle\boldsymbol{\xi}\rangle=0,\qquad\forall\,|\boldsymbol{\xi}|>\delta$$

where δ is the material's horizon in peridynamic theory and $\mathcal{H}=\{\mathbf{Q}|\mathbf{X}\in\mathcal{B},\mathbf{Q}\in\mathcal{B},|\boldsymbol{\xi}|<\delta\}$ is identified as the family of $\mathbf{X}$. Then equation (4.13) and (4.14) can

be written as

$$\begin{aligned}\rho_0^s\ddot{\mathbf{u}}^s =&\rho_0^s\mathbf{G}+\mathbf{H}^s+\int_{\mathcal{H}}\left(\lambda^s(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-\lambda^s(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)\,d\boldsymbol{\xi}+pJ\underline{\nabla}_{\mathbf{x}}\phi^s,\\ &-\int_{\mathcal{H}}\underline{\nabla}\Psi^s(\mathbf{Q},\overline{\rho}^s(\mathbf{Q}))\langle-\boldsymbol{\xi}\rangle-\underline{\nabla}\Psi^s(\mathbf{X},\overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\,d\boldsymbol{\xi},\end{aligned}\tag{4.19}$$

$$\rho_0^f\ddot{\mathbf{u}}^f =\rho_0^f\mathbf{G}+\mathbf{H}^f+\int_{\mathcal{H}}\left(\lambda^f(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-\lambda^f(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)\,d\boldsymbol{\xi}+pJ\underline{\nabla}_{\mathbf{x}}\phi^f.\tag{4.20}$$

Eliminating λ^s and λ^f gives

$$\begin{aligned}\rho_0^s\ddot{\mathbf{u}}^s =&\rho_0^s\mathbf{G}+\mathbf{H}^s+\int_{\mathcal{H}}\left(p(\mathbf{Q})\phi^s(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-p(\mathbf{X})\phi^s(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)\,d\boldsymbol{\xi}\\ &+pJ\underline{\nabla}_{\mathbf{x}}\phi^s-\int_{\mathcal{H}}\underline{\nabla}\Psi^s(\mathbf{Q},\overline{\rho}^s(\mathbf{Q}))\langle-\boldsymbol{\xi}\rangle-\underline{\nabla}\Psi^s(\mathbf{X},\overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\,d\boldsymbol{\xi},\end{aligned}\tag{4.21a}$$

$$\begin{aligned}\rho_0^f\ddot{\mathbf{u}}^f =&\rho_0^f\mathbf{G}+\mathbf{H}^f+\int_{\mathcal{H}}\left(p(\mathbf{Q})\phi^f(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-p(\mathbf{X})\phi^f(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)\,d\boldsymbol{\xi}\\ &+pJ\underline{\nabla}_{\mathbf{x}}\phi^f,\end{aligned}\tag{4.21b}$$

$$p=\frac{(\bar{\rho}^s)^2}{\rho_0^s}\cdot\frac{\partial\Psi^s(\mathbf{X},\overline{\rho}^s)}{\partial\overline{\rho}^s}=\frac{(\bar{\rho}^f)^2}{\rho_0^f}\cdot\frac{\partial\Psi^f(\mathbf{X},\overline{\rho}^f)}{\partial\overline{\rho}^f}.\tag{4.21c}$$

However, there are still three terms $\underline{\nabla}\Psi^s$, $\underline{\nabla}J$ and $\underline{\nabla}_{\mathbf{x}}\phi^\alpha$ that remain unknown in equations (4.21) . In the next chapter, we will present our peridynamic constitutive model based on which these three terms are able to be evaluated nonlocally.

Chapter 5

Peridynamic Constitutive Models for Finite Deformation

One of the original material models proposed for state-based peridynamics is the so-called constitutive correspondence formulation. A peridynamic material model is said to correspond to a classical material model when the strain energy density of both the classical and peridynamic material are equal under affine deformation [Tupek, 2014]. For this purpose, an approximate deformation gradient $\bar{\mathbf{F}}$ was introduced by S. A. Silling in 2007, which has already been mentioned in Chapter 2.2.2 as (2.16) :

$$\bar{\mathbf{F}}(\underline{\mathbf{Y}}) = \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{\mathbf{Y}} \otimes \boldsymbol{\xi}\, \mathrm{d}\boldsymbol{\xi}\right] \cdot \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \boldsymbol{\xi} \otimes \boldsymbol{\xi}\, \mathrm{d}\boldsymbol{\xi}\right]^{-1}.$$

However, peridynamic constitutive models based on this approximate deformation gradient $\bar{\mathbf{F}}$ will inevitably have instabilities due to the fact that the definition of $\bar{\mathbf{F}}$ does not prevent matter interpenetration, namely:

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \to \mathbf{0}, \quad \text{for some } \boldsymbol{\xi} \in \mathcal{H} \not\Longrightarrow \Psi(\bar{\mathbf{F}}(\underline{\mathbf{Y}})) \to \infty;$$

for an in-depth description, refer to Tupek [2014, chap. 4]. Therefore, Tupek proposed the extended constitutive correspondence formulation expressed in terms of generalized nonlocal Seth-Hill strain tensors in 2013. Inspired by his idea, we also

start from the generalized Seth-Hill family of strain measures $\{\mathbf{E}_{(m)}\big|m \in \mathbb{R}\}$, i.e.

$$\mathbf{E}_{(m)} = \frac{1}{2m}(\mathbf{U}^{2m} - \mathbf{I}) = \frac{1}{2m}(\mathbf{C}^{m} - \mathbf{I}),$$

where tensor $\mathbf{U}$ is the right stretch tensor, $\mathbf{C}$ is the right Cauchy-Green deformation tensor, and $\mathbf{I}$ is the identity tensor.

5.1 Peridynamic Right Cauchy-Green Deformation tensor

We define our generalized peridynamic right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$ based on a brand new fourth-order shape tensor:

$$\bar{\mathbf{C}} = \left(\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^2 \boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right) : \bar{\mathbb{L}}, \tag{5.1}$$

or using indicial notation:

$$\bar{C}_{ij} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^2 \xi_m\xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\,\bar{L}_{mnij}, \tag{5.2}$$

where $\bar{\mathbb{L}}$ is a fourth order peridynamic tensor and the inverse of the peridynamic fourth-order shape tensor $\bar{\mathbb{K}}$ defined as using indicial notation

$$\bar{K}_{ijkl} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_i\xi_j\xi_k\xi_l}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}, \tag{5.3}$$

which, because of the symmetry of the fourth order tensor, satisfy

$$\bar{K}_{ijkl}\bar{L}_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \tag{5.4}$$

Here, we first prove the correspondence of our peridynamic deformation tensor $\bar{\mathbf{C}}$ to the classical right Cauchy-Green deformation tensor $\mathbf{C}$ when the deformation is homogeneous. If the deformation is homogeneous, we can express the deformation vector-state $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$ as

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi}, \tag{5.5}$$

where $\mathbf{F}$ is the classical deformation gradient. Substituting (5.5) into (5.2) gives

$$\begin{aligned}
\bar{C}_{ij} =& \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^2 \xi_m \xi_n}{|\boldsymbol{\xi}|^4} \, \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnij}, \\
=& \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{F_{kp}\xi_p F_{kq}\xi_q \xi_m \xi_n}{|\boldsymbol{\xi}|^4} \, \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnij}, \\
=& F_{kp}F_{kq} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_p \xi_q \xi_m \xi_n}{|\boldsymbol{\xi}|^4} \, \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnij}, \\
=& F_{kp}F_{kq}\bar{K}_{pqmn}\bar{L}_{mnij}, \\
=& F_{ki}F_{kj}, \\
=& C_{ij},
\end{aligned} \tag{5.6}$$

which concludes that the peridynamic deformation tensor is identical to the classical right Cauchy-Green deformation tensor $\mathbf{C}$ when the deformation is homogeneous.

Given the fact that the length of the bond $|\boldsymbol{\xi}|$ becomes the length of the deformation vector-state $|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|$ after a deformation described by peridynamic deformation tensor $\bar{\mathbf{C}}$, we can also choose the extension of bond length $\frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}{|\boldsymbol{\xi}|}$ as an indicator of the deformation. Then it is reasonable to propose that the extension of bond length will be $\left(\frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}{|\boldsymbol{\xi}|}\right)^m$ after the deformation to the power of m whose peridynamic right Cauchy-Green deformation tensor is denoted by $\bar{\mathbf{C}}^{(m)}$. Since our peridynamic deformation tensor defined as (5.1) is only dependent on the length of deformation vector-state, the peridynamic deformation tensor $\bar{\mathbf{C}}^{(m)}$ describing the deformation to the power of m can be expressed as

$$\bar{\mathbf{C}}^{(m)} = \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(|\boldsymbol{\xi}| \cdot \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^m\right)^2 \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^4} \, \mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}}. \tag{5.7}$$

5.2 Peridynamic Strain Family

With the definition of $\bar{\mathbf{C}}^{(m)}$, we are able to define our peridynamic Seth-Hill strain family $\bar{\mathbf{E}}_{(m)}$ as

$$\bar{\mathbf{E}}_{(m)} = \frac{1}{2m}\left(\bar{\mathbf{C}}^{(m)} - \mathbf{I}\right). \tag{5.8}$$

As a remark, the peridynamic logarithmic strain measure is just $\lim_{m\to 0}\bar{\mathbf{E}}_{(m)}$ according to (5.8) and it can be simplified as showed by following equations. Series expansion of (5.7) about $m = 0$ gives

$$\bar{\mathbf{C}}^{(0)} = \int_{\mathcal{H}} \left(\underline{\omega}\langle \boldsymbol{\xi} \rangle + 2m\, \underline{\omega}\langle \boldsymbol{\xi} \rangle \ln\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) + \mathcal{O}(|m|^2) \right) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi}. \tag{5.9}$$

Substituting (5.9) into (5.8) and taking the limit as $m \to 0$ gives

$$\begin{aligned}
\bar{\mathbf{E}}_{(0)} =& \frac{1}{2m} \left(\int_{\mathcal{H}} \left(\underline{\omega}\langle \boldsymbol{\xi} \rangle + 2m\, \underline{\omega}\langle \boldsymbol{\xi} \rangle \ln\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) + \mathcal{O}(|m|^2) \right) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi} - \mathbf{I} \right), \\
=& \frac{1}{2m} \left(\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi} - \mathbf{I} \right) + \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \ln\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi} \\
&+ \frac{1}{2m} \int_{\mathcal{H}} \mathcal{O}(|m|^2) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi}, \\
=& \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \ln\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \,:\, \bar{\mathbb{L}}\, \mathrm{d}\boldsymbol{\xi},
\end{aligned} \tag{5.10}$$

where the last term in the second line vanishes using an implicit invocation of L'Hôpital's rule and the first term vanishes via the following proof

$$\begin{aligned}
\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_m \xi_n |\boldsymbol{\xi}|^2}{|\boldsymbol{\xi}|^4}\, \mathrm{d}\boldsymbol{\xi} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}, \\
\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_l \xi_l \xi_m \xi_n}{|\boldsymbol{\xi}|^4}\, \mathrm{d}\boldsymbol{\xi} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}, \\
\bar{K}_{llmn} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi} \cdot \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \\
\delta_{kl}\bar{K}_{klmn} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}\, \bar{L}_{ijkl}\bar{K}_{klmn}, \\
\delta_{kl}\bar{K}_{klmn}\bar{L}_{mnpq} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}\, \bar{L}_{ijkl}\bar{K}_{klmn}\bar{L}_{mnpq}, \\
\delta_{kl} \cdot \frac{1}{2}(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}) &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}\, \bar{L}_{ijkl} \cdot \frac{1}{2}(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}), \\
\delta_{pq} &= \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\, \mathrm{d}\boldsymbol{\xi}\, \bar{L}_{ijpq}.
\end{aligned}$$

5.2.1 Correspondence to infinitesimal strain

In this section, we will prove the correspondence of our peridynamic Seth-Hill strain family $\bar{\mathbf{E}}_{(m)}$ to the classical infinitesimal strain. In this case that the strain is infinitesimal, the deformation vector-state can be expressed as

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} + \nabla\mathbf{u}\cdot\boldsymbol{\xi},$$

or using indicial notation

$$\underline{Y}_i = \xi_i + \frac{\partial u_i}{\partial \xi_j}\xi_j.$$

Then, $|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^2$ can be written as ignoring high order terms of $\frac{\partial u_i}{\partial \xi_j}$:

$$|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^2 = \xi_i\xi_i + 2\frac{\partial u_i}{\partial \xi_j}\xi_i\xi_j. \tag{5.11}$$

Substituting (5.11) into (5.7) and using indicial notation, we have

$$
\begin{aligned}
\bar{C}_{ij}^{(m)} &= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(|\boldsymbol{\xi}| \cdot \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^m\right)^2 \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|^2}{|\boldsymbol{\xi}|^2}\right)^m \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{\xi_k\xi_k + 2\frac{\partial u_k}{\partial \xi_l}\xi_k\xi_l}{|\boldsymbol{\xi}|^2}\right)^m \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(1 + 2\frac{\frac{\partial u_k}{\partial \xi_l}\xi_k\xi_l}{|\boldsymbol{\xi}|^2}\right)^m \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(1 + 2m\frac{\frac{\partial u_k}{\partial \xi_l}\xi_k\xi_l}{|\boldsymbol{\xi}|^2}\right) \frac{xi_m \xi_n}{|\boldsymbol{\xi}|^2}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij} + \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\, 2m\frac{\partial u_k}{\partial \xi_l}\frac{\xi_k\xi_l\xi_m\xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \bar{K}_{kkmn}\bar{L}_{mnij} + 2m\frac{\partial u_k}{\partial \xi_l}\left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_k\xi_l\xi_m\xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right] \bar{L}_{mnij}, \\
&= \bar{K}_{kkmn}\bar{L}_{mnij} + 2m\frac{\partial u_k}{\partial \xi_l}\bar{K}_{klmn}\bar{L}_{mnij}, \\
&= \delta_{ij} + 2m\frac{\partial u_k}{\partial \xi_l}\cdot\frac{1}{2}(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}), \\
&= \delta_{ij} + m\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right).
\end{aligned} \tag{5.12}
$$

Substitute (5.12) into (5.8) using indicial notation and our peridynamic strain tensor $\bar{\mathbf{E}}_{(m)}$ just becomes

$$
\begin{aligned}
\bar{E}_{(m)ij} &= \frac{1}{2m}\left(\bar{C}_{ij}^{(m)} - \delta_{ij}\right), \\
&= \frac{1}{2m}\left(\delta_{ij} + m\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right) - \delta_{ij}\right), \\
&= \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right).
\end{aligned} \tag{5.13}
$$

Therefore, we conclude that in the limit of uniform infinitesimal strains, the peridynamic strain tensor is identical to the classical strain tensor.

5.2.2 Correspondence for Pure expansion

Since the idea of proposing (5.7) is based on the extension of bond length, the correspondence of $\bar{\mathbf{E}}_{(m)}$ to classical Seth-Hill strain family is quite straightforward when deformation is pure expansion except for logarithmic strain $\bar{\mathbf{E}}_{(0)}$. Therefore, in this section, we show that $\bar{\mathbf{E}}_{(0)}$ is exactly the log strain tensor fro large uniform pure expansions. For the special case $m = 0$, $\bar{\mathbf{E}}_{(m)}$ becomes

$$\bar{\mathbf{E}}_{(m)} = \frac{1}{2} ln \bar{\mathbf{C}} = \left[\int_{\mathcal{H}} \underline{\omega} \langle \boldsymbol{\xi} \rangle \left(|\boldsymbol{\xi}| \cdot ln \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) \right)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi} / |\boldsymbol{\xi}|^4 \, \mathrm{d}\boldsymbol{\xi} \right] : \bar{\mathbb{L}}.$$

Consider a uniform volumetric strain with deformation gradient $\mathbf{F} = J^{\frac{1}{3}} \mathbf{R}$. The peridynamic deformation vector-state is

$$\underline{\mathbf{Y}} \langle \boldsymbol{\xi} \rangle = \mathbf{F} \boldsymbol{\xi} = J^{\frac{1}{3}} \mathbf{R} \boldsymbol{\xi}, \tag{5.14}$$

where $\mathbf{R}$ is a rotation tensor, $\mathbf{R}\mathbf{R}^{\intercal} = \mathbf{R}^{\intercal}\mathbf{R} = \mathbf{1}$, and J represents the volume change between reference and deformed configuration. The actual (classical) logarithmic strain tensor $\mathbf{E}_{(0)}$ for this deformation gradient is

$$\mathbf{E}_{(0)} = \frac{1}{2} \log(\mathbf{F}^{\intercal}\mathbf{F}) = \frac{1}{2} \log(J^{\frac{2}{3}} \mathbf{R}^{\intercal}\mathbf{R}) = \frac{1}{3} \log(J) \mathbf{I}. \tag{5.15}$$

On the other hand, substituting (5.14) into equation (5.10) gives

$$\begin{aligned} \bar{E}_{(0)ij} &= \left[\int_{\mathcal{H}} \underline{\omega} \langle \boldsymbol{\xi} \rangle \left(\ln \frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|} \right) \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2} \, \mathrm{d}\boldsymbol{\xi} \right] \bar{L}_{mnij}, \\ &= \left[\int_{\mathcal{H}} \underline{\omega} \langle \boldsymbol{\xi} \rangle ln \left(J^{\frac{1}{3}} \right) \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2} \, \mathrm{d}\boldsymbol{\xi} \right] \bar{L}_{mnij}, \\ &= \frac{1}{3} ln J \cdot \bar{K}_{kkmn} \bar{L}_{mnij}, \\ &= \frac{1}{3} ln J \cdot \delta_{ij}. \end{aligned} \tag{5.16}$$

Comparing (5.15) with (5.16) concludes that $\bar{\mathbf{E}}_{(0)}$ is identical to classical logarithmic strain $\mathbf{E}_{(0)}$ when the deformation is pure volumetric expansion.

5.2.3 Simplification for Isotropic Material

In order to show the applicability of our peridynamic strain tensor family $\bar{\mathbf{E}}_{(m)}$ based on fourth-order peridynamic shape tensor $\bar{\mathbb{K}}$, we will simplify our peridynamic constitutive models under the assumption that the material is isotropic in this section. If the isotropy of the material is assumed, the peridynamic integrand will be symmetric and influence function $\underline{\omega}\langle\boldsymbol{\xi}\rangle$ will be spherical, i.e.

$$\underline{\omega}\langle\boldsymbol{\xi}\rangle = \underline{\omega}\langle|\boldsymbol{\xi}|\rangle. \tag{5.17}$$

Substituting (5.17) into (5.3) gives

$$\begin{aligned}\bar{K}_{ijkl} =& \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_i\xi_j\xi_k\xi_l}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi},\\ =&(\delta_{ij}\delta_{kl}(1-\delta_{ik})+\delta_{ik}\delta_{jl}(1-\delta_{ij})+\delta_{il}\delta_{jk}(1-\delta_{ij}))\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_1^2\xi_2^2}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\\ &+\delta_{ij}\delta_{jk}\delta_{kl}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_1^4}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi},\end{aligned} \tag{5.18}$$

where the symmetry of the integrand has been exploited to evaluate only the nonzero combination of indices in the integrals as shown in the second of (5.18) . If we define the weighted volume M as

$$M = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\,\mathrm{d}\boldsymbol{\xi},$$

then [Silling et al., 2007]

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_1^4}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi} = \frac{M}{5}, \qquad \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_1^2\xi_2^2}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi} = \frac{M}{15}. \tag{5.19}$$

Substituting (5.19) into (5.18) , the shape tensor $\bar{K}_{ijkl}$ can be simplified as

$$\bar{K}_{ijkl} = \frac{M}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{5.20}$$

Since the isotropic fourth-order shape tensor's inverse $\mathbb{L}$ is also isotropic, we can write it as

$$\bar{L}_{mnkl} = A\delta_{mn}\delta_{kl} + B(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{nk}), \tag{5.21}$$

where A and B are unknown constants. Substituting (5.20) and (5.21) into (5.4) gives

$$\begin{aligned}
\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) =& \bar{K}_{ijmn}\bar{L}_{mnkl},\\
\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) =& \frac{M}{15}\left(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\right)\left(A\delta_{mn}\delta_{kl}+B\delta_{mk}\delta_{nl}+B\delta_{ml}\delta_{nk}\right),\\
\frac{15}{2M}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) =& A\left(3\delta_{ij}\delta_{kl}+\delta_{ij}\delta_{kl}+\delta_{ij}\delta_{kl}\right)+2B\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}\right),\\
\frac{15}{2M}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) =& 5A\delta_{ij}\delta_{kl}+2B\delta_{ij}\delta_{kl}+2B\left(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}\right).
\end{aligned}$$

Comparing the coefficients of each terms, we have

$$\begin{aligned}
5A+2B&=0,\\
B&=\frac{15}{4M}.
\end{aligned}$$

so

$$A = -\frac{3}{2M}, \tag{5.22a}$$

$$B = -\frac{15}{4M}. \tag{5.22b}$$

Substituting (5.22) into (5.21) gives us simplification of the inverse of isotropic fourth-order shape tensor:

$$\bar{L}_{mnkl} = \frac{15}{4M}(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{nk}) - \frac{3}{2M}\delta_{mn}\delta_{kl}. \tag{5.23}$$

Substituting (5.23) into (5.7) using indicial notation gives

$$\begin{aligned}
\bar{C}_{ij}^{(m)} &= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\right] \bar{L}_{mnij}, \\
&= \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \frac{\xi_m \xi_n}{|\boldsymbol{\xi}|^2}\right] \left[\frac{15}{4M}(\delta_{mi}\delta_{nj} + \delta_{mj}\delta_{ni}) - \frac{3}{2M}\delta_{mn}\delta_{ij}\right], \\
&= \frac{15}{2M} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} \,\mathrm{d}\boldsymbol{\xi} - \frac{3\delta_{ij}}{2M} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \mathrm{d}\boldsymbol{\xi}, \\
&= \frac{3}{M} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \left(\frac{5}{2}\frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} - \frac{1}{2}\delta_{ij}\right) \mathrm{d}\boldsymbol{\xi}. \qquad (5.24)
\end{aligned}$$

Using the notation $\bar{\mathbf{H}}$ for a peridynamic tensor defined as

$$\bar{\mathbf{H}} = \frac{3}{M}\left(\frac{5}{2}\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} - \frac{1}{2}\mathbf{I}\right),$$

equation (5.24) becomes

$$\bar{\mathbf{C}}_{(m)} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \bar{\mathbf{H}} \,\mathrm{d}\boldsymbol{\xi},$$

and the peridynamic tensor $\bar{\mathbf{H}}$ is identical to the shape tensor-state used by Tupek [2014], which here turns out to be a simplification for isotropic materials from a more generalized fourth-order shape tensor $\bar{\mathbb{K}}$.

5.2.4 Poroelasticity Peridynamic Modeling

As an example of our peridynamic constitutive models' application, in this section, we will evaluate the Fréchet derivative of solid internal energy $\underline{\nabla}\Psi^s$ that appears in the peridynamic governing equations for poroelasticity (4.21) based on our peridynamic deformation tensor-state $\bar{\mathbf{C}}^{(m)}$ along with semilinear approximation (Berryman and Thigpen, 1984).

First, we calculate the first variation of the peridynamic deformation tensor-

state $\bar{\mathbf{C}}^{(m)}$ using (5.7) :

$$\begin{aligned}
\bar{\mathbf{C}}^{(m)} =& \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}},\\
\delta\bar{\mathbf{C}}^{(m)} =& \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \cdot \delta\left(\frac{\underline{\mathbf{Y}}\cdot\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right)^{m} \boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}},\\
=& m\left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{\underline{\mathbf{Y}}\cdot\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right)^{m-1} \cdot \delta\left(\frac{\underline{\mathbf{Y}}\cdot\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right) \boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}},\\
=& 2m\left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \cdot \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}}.
\end{aligned}$$

Along with the definition of Fréchet derivative (2.11) , we can write the Fréchet derivate of the peridynamic deformation tensor $\underline{\nabla}\bar{\mathbf{C}}^{(m)}$ as

$$\underline{\nabla}\bar{\mathbf{C}}^{m} = 2m\left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \cdot \frac{\underline{\mathbf{Y}}\otimes\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^4} \,\mathrm{d}\boldsymbol{\xi}\right] : \bar{\mathbb{L}}. \tag{5.25}$$

For simplicity of the following derivation, we will assume the solid material is isotropic and the integrand is symmetric. Then (5.25) can be simplified to

$$\underline{\nabla}\bar{\mathbf{C}}^{m} = 2m\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \cdot \frac{\underline{\mathbf{Y}}\otimes\bar{\mathbf{H}}}{|\boldsymbol{\xi}|^2} \,\mathrm{d}\boldsymbol{\xi}. \tag{5.26}$$

Similarly, the first variations of the traces of the peridynamic deformation tensor

$\delta\mathrm{tr}(\bar{\mathbf{C}}^{(m)})$, $\delta\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^2\right)$ and $\delta\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^3\right)$ can be written as

$$
\begin{aligned}
\delta tr(\bar{\mathbf{C}}^{(m)}) =& \delta \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \mathrm{d}\boldsymbol{\xi}, \\
=& \delta \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{\underline{\mathbf{Y}}\cdot\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right)^{m} \mathrm{d}\boldsymbol{\xi}, \\
=& 2m \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{\underline{\mathbf{Y}}\cdot\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right)^{m-1} \cdot \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}, \\
=& 2m \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}; \qquad (5.27a) \\
\delta\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^2\right) =& \delta\left(\bar{C}_{ij}\bar{C}_{ji}\right), \\
=& 2\bar{C}_{ij}\cdot\delta\bar{C}_{ij}, \\
=& 2\bar{C}_{ij}\cdot\delta\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m} \bar{H}_{ij} \mathrm{d}\boldsymbol{\xi}, \\
=& 4m\bar{C}_{ij}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \bar{H}_{ij} \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}, \\
=& 4m\cdot tr\left(\bar{\mathbf{C}}_{(m)}\bar{\mathbf{H}}\right) \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}; \qquad (5.27b) \\
\delta\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^3\right) =& \delta\left(\bar{C}_{ij}\bar{C}_{jk}\bar{C}_{ki}\right), \\
=& 3\bar{C}_{ij}\bar{C}_{jk}\cdot\delta\bar{C}_{ki}, \\
=& 6m\bar{C}_{ij}\bar{C}_{jk}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \bar{H}_{ki} \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}, \\
=& 6m\cdot\mathrm{tr}\left(\bar{\mathbf{C}}^2_{(m)}\bar{\mathbf{H}}\right) \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\cdot \left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2} \frac{\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} \mathrm{d}\boldsymbol{\xi}. \qquad (5.27c)
\end{aligned}
$$

Substituting (5.27) into the relations between invariants and traces:

$$
\begin{aligned}
\bar{I}_1 =& \mathrm{t}r(\bar{\mathbf{C}}^{(m)}), \\
\bar{I}_2 =& \frac{1}{2}\left(tr(\bar{\mathbf{C}}^{(m)})^2 - \mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^2\right)\right), \\
\bar{I}_3 =& \frac{1}{6}\left(tr(\bar{\mathbf{C}}^{(m)})^3 - 3\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^2\right)\mathrm{tr}(\bar{\mathbf{C}}^{(m)}) + 2\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^3\right)\right), \qquad (5.28)
\end{aligned}
$$

where $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ are three invariants of $\bar{\mathbf{C}}^{(m)}$; along with the definition of Fréchet

derivative (2.10) gives us the results of Fréchet derivatives of the three invariants:

$$\underline{\nabla}\bar{I}_1 = 2m\cdot\underline{\omega}\langle\boldsymbol{\xi}\rangle\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2}\frac{\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}; \tag{5.29a}$$

$$\begin{aligned}\underline{\nabla}\bar{I}_2 =& 2m\cdot\underline{\omega}\langle\boldsymbol{\xi}\rangle\, tr(\bar{\mathbf{C}}^{(m)})\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2}\frac{\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2} - 2m\cdot\underline{\omega}\langle\boldsymbol{\xi}\rangle\, tr\left(\bar{\mathbf{C}}^{(m)}\bar{\mathbf{H}}\right)\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2}\frac{\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2},\\ =& 2m\cdot\underline{\omega}\langle\boldsymbol{\xi}\rangle\,\left(\mathrm{tr}(\bar{\mathbf{C}}^{(m)}) - \mathrm{tr}\left(\bar{\mathbf{C}}^{(m)}\bar{\mathbf{H}}\right)\right)\left(\frac{|\underline{\mathbf{Y}}|}{|\boldsymbol{\xi}|}\right)^{2m-2}\frac{\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2},\\ =& \left(\bar{I}_1 - \mathrm{tr}\left(\bar{\mathbf{C}}^{(m)}\bar{\mathbf{H}}\right)\right)\underline{\nabla}\bar{I}_1;\end{aligned} \tag{5.29b}$$

$$\begin{aligned}\underline{\nabla}\bar{I}_3 =& \frac{1}{6}\Big(3\mathrm{tr}(\bar{\mathbf{C}}^{(m)})^2\cdot\underline{\nabla}\bar{I}_1 - 3\mathrm{tr}\left((\bar{\mathbf{C}}^{(m)})^2\right)\cdot\underline{\nabla}\bar{I}_1\\ &\qquad -6tr(\bar{\mathbf{C}}^{(m)})\cdot tr(\bar{\mathbf{C}}^{(m)}\bar{\mathbf{H}})\cdot\underline{\nabla}\bar{I}_1 + 6\mathrm{tr}\left(\left(\bar{\mathbf{C}}^{(m)}\right)^2\bar{\mathbf{H}}\right)\cdot\underline{\nabla}\bar{I}_1\Big).\\ =& \left(\bar{I}_2 - \bar{I}_1\cdot tr(\bar{\mathbf{C}}^{(m)}\bar{\mathbf{H}}) + \mathrm{tr}\left(\left(\bar{\mathbf{C}}^{(m)}\right)^2\bar{\mathbf{H}}\right)\right)\underline{\nabla}\bar{I}_1.\end{aligned} \tag{5.29c}$$

On the other hand, the semilinear approximation in classical theory states that the internal energy of the solid can be expressed in terms of three invariants of strain tensor [Berryman and Thigpen, 1985]:

$$\begin{aligned}\Psi^s =& \frac{1}{2}aI_1^{s2} + bI_2^s + cI_1^s(\overline{\rho}^s - \overline{\rho}_0^s) + \frac{1}{2}d(\overline{\rho}^s - \overline{\rho}_0^s)^2 + eI_1^{s3} + fI_1^sI_2^s + gI_3^s\\ &+ mI_1^{s2}(\overline{\rho}^s - \overline{\rho}_0^s) + nI_2^s(\overline{\rho}^s - \overline{\rho}_0^s),\end{aligned} \tag{5.30}$$

$$\Psi^f = \frac{1}{2}h(\overline{\rho}^f - \overline{\rho}_0^f)^2, \tag{5.31}$$

where Ψ^s and Ψ^f are the internal energy densities of solid and fluid; I_i^s $(i = 1, 2, 3)$ are the invariants of Lagrangian strain tensor for solid; $\overline{\rho}_0^s$ and $\overline{\rho}_0^f$ are the initial local densities of solid and fluid; $\overline{\rho}^s$ and $\overline{\rho}^f$ are the local densities of solid and fluid that varies throughout the motion; and the rest of coefficients are material constants. Here,we also apply the semilinear approximation and express the solid energy density in terms of three invariants of the peridynamic deformation tensor, $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$:

$$\begin{aligned}\Psi^s =& \frac{1}{2}a\bar{I}_1^2 + b\bar{I}_2 + c\bar{I}_1(\overline{\rho}^s - \overline{\rho}_0^s) + \frac{1}{2}d(\overline{\rho}^s - \overline{\rho}_0^s)^2 + e\bar{I}_1^3 + f\bar{I}_1\bar{I}_2 + g\bar{I}_3\\ &+ m\bar{I}_1^2(\overline{\rho}^s - \overline{\rho}_0^s) + n\bar{I}_2(\overline{\rho}^s - \overline{\rho}_0^s).\end{aligned} \tag{5.32}$$

Therefore, the Fréchet derivative of solid internal energy density $\underline{\nabla}\Psi^s$ is

$$\begin{aligned}\underline{\nabla}\Psi^s =& a\bar{I}_1\underline{\nabla}\bar{I}_1 + b\underline{\nabla}\bar{I}_2 + c(\overline{\rho}^s - \overline{\rho}_0^s)\underline{\nabla}\bar{I}_1 + 3e\bar{I}_1^2\underline{\nabla}\bar{I}_1 + f\bar{I}_1\underline{\nabla}\bar{I}_2 + f\bar{I}_2\underline{\nabla}\bar{I}_1 + g\underline{\nabla}\bar{I}_3 \\ &+ 2mI_1^s(\overline{\rho}^s - \overline{\rho}_0^s)\underline{\nabla}\bar{I}_1 + n(\overline{\rho}^s - \overline{\rho}_0^s)\underline{\nabla}\bar{I}_2^s, \\ =& \left(a\bar{I}_1 + c(\overline{\rho}^s - \overline{\rho}_0^s) + 3e\bar{I}_1^2 + f\bar{I}_2 + 2mI_1^s(\overline{\rho}^s - \overline{\rho}_0^s)\right)\underline{\nabla}\bar{I}_1^s \\ &+ \left(b + f\bar{I}_1 + n(\overline{\rho}^s - \overline{\rho}_0^s)\right)\underline{\nabla}\bar{I}_2 + g\underline{\nabla}\bar{I}_3. \end{aligned} \tag{5.33}$$

Substituting (5.29) into (5.33) gives the final result of the Fréchet derivative of solid internal energy density $\underline{\nabla}\Psi^s$ expressed in terms of peridynamic deformation tensor-state $\bar{\mathbf{C}}^{(m)}$, which is too lengthy for us to show the final equation.

5.3 Peridynamic Model for Volume Ratio

Since section 5.1 has already defined our peridynamic right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$ and shown its correspondence to the classical deformation tensor, we can just evaluate the volume ratio J using $\bar{\mathbf{C}}$ as classical right Cauchy-Green deformation tensor $\mathbf{C}$, based on the relation:

$$J^2 = \det(\mathbf{C}). \tag{5.34}$$

However, relation (5.34) is a result of classical continuum mechanics theory where the deformation is assumed to be homogeneous, so this relation no longer holds for peridynamic theory, which raise doubts about substituting peridynamic deformation tensor-state $\bar{\mathbf{C}}$ into (5.34) to evaluate volume ration. Therefore, in this section, we introduce a new peridynamic model for evaluate volume ratio J nonlocally based on its physical meaning.

Volume ratio J describes the ratio between the volume after deformation and the original volume in the reference configuration. Consider the parallelepiped defined by three bonds $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the reference configuration. Its signed volume $V_0(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ equals to the scalar triple product of these three vectors:

$$V_0(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = (\boldsymbol{\xi} \times \boldsymbol{\eta}) \cdot \boldsymbol{\zeta}.$$

After deformations, these three vectors deform into $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$, $\underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle$ and $\underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle$. Therefore

its deformed volume $V_d(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ is

$$V_d(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = (\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle) \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle.$$

The volume ratio of the parallelepiped $J_V(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ can be written as

$$J_V^2 = \frac{((\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle) \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle)^2}{((\boldsymbol{\xi} \times \boldsymbol{\eta}) \cdot \boldsymbol{\zeta})^2},$$

which is expressed in square form in order to get rid of the signs of signed volumes V_0 and V_d. Based on this idea, one way to describe the volume ratio of the family $\mathcal{H}$ as a whole is to integrate all the volume ratios of every parallelepiped defined by every three bond vectors inside the family $\mathcal{H}$, which can be written as

$$J^2 = \frac{\int_{\mathcal{H}}\int_{\mathcal{H}}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle \left((\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle) \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle\right)^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}}{\int_{\mathcal{H}}\int_{\mathcal{H}}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle \left((\boldsymbol{\xi} \times \boldsymbol{\eta}) \cdot \boldsymbol{\zeta}\right)^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}}. \tag{5.35}$$

While the physical justification for this nonlocal model of volume ratio should be clear, the practical application of it, in a computational setting would be cumbersome in its current form. Therefore, in what follows, we will demonstrate that the right side of (5.35) can be shown to be the determinant of a peridynamic tensor-state $\bar{\mathbf{J}}$, which provides a much easier way to compute.

We define the peridynamic tensor $\bar{\mathbf{J}}$ as

$$\bar{\mathbf{J}} := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \otimes \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \,\mathrm{d}\boldsymbol{\xi} \cdot \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi} \otimes \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{\xi}\right]^{-1}, \tag{5.36}$$

which can be viewed as a peridynamic tensor $\bar{\mathbf{A}}$ divided by a peridynamic tensor $\bar{\mathbf{D}}$ which are defined as

$$\bar{\mathbf{A}} := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{Y}} \otimes \underline{\mathbf{Y}} \,\mathrm{d}\boldsymbol{\xi},$$
$$\bar{\mathbf{D}} := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi} \otimes \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{\xi}\ .$$

Then, the determinant of the peridynamic tensor $\bar{\mathbf{A}}$ can be written as

$$
\begin{aligned}
\det(\bar{\mathbf{A}}) &= \frac{1}{6}(\mathrm{tr}(\bar{\mathbf{A}})^3 - 3\mathrm{tr}(\bar{\mathbf{A}}^2)\mathrm{tr}(\bar{\mathbf{A}}) + 2\mathrm{tr}(\bar{\mathbf{A}}^3)), \\
&= \frac{1}{6}(\bar{A}_{qq}\bar{A}_{oo}\bar{A}_{ii} - \bar{A}_{oq}\bar{A}_{qo}\bar{A}_{pp} - \bar{A}_{oo}\bar{A}_{pq}\bar{A}_{qp} \\
&\qquad + \bar{A}_{oq}\bar{A}_{qp}\bar{A}_{po} + \bar{A}_{pq}\bar{A}_{qo}\bar{A}_{op} - \bar{A}_{qq}\bar{A}_{po}\bar{A}_{op}), \\
&= \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_i\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_i\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta} \\
&\quad - \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_o\langle\boldsymbol{\xi}\rangle\underline{Y}_q\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta} \\
&\quad - \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_p\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_q\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta} \\
&\quad + \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_o\langle\boldsymbol{\xi}\rangle\underline{Y}_p\langle\boldsymbol{\eta}\rangle\underline{Y}_q\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta} \\
&\quad + \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_p\langle\boldsymbol{\xi}\rangle\underline{Y}_q\langle\boldsymbol{\eta}\rangle\underline{Y}_o\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta} \\
&\quad - \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_p\langle\boldsymbol{\eta}\rangle\underline{Y}_o\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}, \\
&= \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle\epsilon_{ijk}\epsilon_{pqo}\underline{Y}_j\langle\boldsymbol{\xi}\rangle\underline{Y}_k\langle\boldsymbol{\eta}\rangle\underline{Y}_i\langle\boldsymbol{\zeta}\rangle\underline{Y}_q\langle\boldsymbol{\xi}\rangle\underline{Y}_o\langle\boldsymbol{\eta}\rangle\underline{Y}_p\langle\boldsymbol{\zeta}\rangle \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}, \\
&= \int_{\mathcal{B}}\int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle \left((\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle)\cdot\underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle\right)^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}, \qquad (5.37)
\end{aligned}
$$

which is identical to the numerator of the right side of (5.35). Similarly, it can also be proved, using the same steps as (5.37), that the determinant of $\bar{\mathbf{D}}$ is equal to the denominator of the right side of (5.35):

$$
\det(\bar{\mathbf{D}}) = \int_{\mathcal{H}}\int_{\mathcal{H}}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\omega}\langle\boldsymbol{\eta}\rangle\underline{\omega}\langle\boldsymbol{\zeta}\rangle \left((\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\zeta}\right)^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\zeta}.
$$

Then, the determinant of the peridynamic tensor $\bar{\mathbf{J}}$ becomes

$$
\det(\bar{\mathbf{J}}) = \frac{\det(\bar{\mathbf{A}})}{\det(\bar{\mathbf{D}})} = J^2.
$$

Hence, we introduce our nonlocal model for the volume ratio J in a physical justified way with a sophisticated equation (5.35), and we are able to compute the nonlocal volume ratio using an easily-compute peridynamic tensor $\bar{\mathbf{J}}$.

If we further assume that the integrand is symmetric and the influence function is spherical, the peridynamic tensor $\bar{\mathbf{J}}$ can be further simplified to

$$\bar{\mathbf{J}} = \frac{1}{K}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \otimes \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \,\mathrm{d}\boldsymbol{\xi},$$

where K is defined as

$$K := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{|\boldsymbol{\xi}|^2}{3}\,\mathrm{d}\boldsymbol{\xi}.$$

Then, following the same steps in section 5.2.4, we calculate the Fréchet derivative of volume ration $\underline{\nabla} J$. First we consider the first variations of the traces of $\bar{\mathbf{J}}$:

$$\begin{aligned}
\delta\mathrm{tr}(\bar{\mathbf{J}}) =& \frac{1}{K}\delta\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle |\underline{\mathbf{Y}}|^2 \,\mathrm{d}\boldsymbol{\xi},\\
=& \frac{2}{K}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}\,\mathrm{d}\boldsymbol{\xi}; \qquad (5.38a)\\
\delta\mathrm{tr}(\bar{\mathbf{J}}^2) =& \delta\left(\bar{J}_{ij}\bar{J}_{ji}\right).\\
=& 2\bar{J}_{ij}\cdot\delta\bar{J}_{ij},\\
=& 2\bar{J}_{ij}\cdot\delta\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\underline{\mathbf{Y}}_i\underline{\mathbf{Y}}_j}{K}\,\mathrm{d}\boldsymbol{\xi},\\
=& 4\bar{J}_{ij}\cdot\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\underline{\mathbf{Y}}_i\cdot\delta\underline{\mathbf{Y}}_j}{K}\,\mathrm{d}\boldsymbol{\xi},\\
=& \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{4\bar{\mathbf{J}}\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{K}\,\mathrm{d}\boldsymbol{\xi}; \qquad (5.38b)\\
\delta\mathrm{tr}(\bar{\mathbf{J}}^3) =& \delta\left(\bar{J}_{ij}\bar{J}_{jk}\bar{J}_{ki}\right),\\
=& 3\bar{J}_{ij}\bar{J}_{jk}\cdot\delta\bar{J}_{ki},\\
=& 6\bar{J}_{ij}\bar{J}_{jk}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\underline{\mathbf{Y}}_k\cdot\delta\underline{\mathbf{Y}}_i}{K}\,\mathrm{d}\boldsymbol{\xi},\\
=& \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{6\bar{\mathbf{J}}^2\underline{\mathbf{Y}}\cdot\delta\underline{\mathbf{Y}}}{K}\,\mathrm{d}\boldsymbol{\xi}.
\end{aligned}$$

Substituting (5.38) into

$$J^2 = \frac{1}{6}\left(\mathrm{tr}(\bar{\mathbf{J}})^3 - 3\mathrm{tr}(\bar{\mathbf{J}}^2)\mathrm{tr}(\bar{\mathbf{J}}) + 2\mathrm{tr}(\bar{\mathbf{J}}^3)\right),$$

we have

$$
\begin{aligned}
2J\underline{\nabla}J =& \frac{\underline{\omega}\langle\boldsymbol{\xi}\rangle}{6}\left(\left(3\mathrm{tr}(\bar{\mathbf{J}})^2-3\mathrm{tr}(\bar{\mathbf{J}}^2)\right)\cdot\frac{2\underline{\mathbf{Y}}}{K}-3\mathrm{tr}(\bar{\mathbf{J}})\cdot\frac{4\bar{\mathbf{J}}\underline{\mathbf{Y}}}{K}+2\cdot\frac{6\bar{\mathbf{J}}^2\underline{\mathbf{Y}}}{K}\right),\\
=&\underline{\omega}\langle\boldsymbol{\xi}\rangle\left(\frac{2\bar{I}_2^J\cdot\underline{\mathbf{Y}}}{K}-\frac{2\bar{I}_1^J\cdot\bar{\mathbf{J}}\underline{\mathbf{Y}}}{K}+\frac{2\bar{\mathbf{J}}^2\underline{\mathbf{Y}}}{K}\right),\\
\underline{\nabla}J =& \underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\left(\bar{I}_2^J\cdot\mathbf{I}-\bar{I}_1^J\cdot\bar{\mathbf{J}}+\bar{\mathbf{J}}^2\right)\underline{\mathbf{Y}}}{KJ},
\end{aligned}
\tag{5.39}
$$

where $\bar{I}_1^J$ and $\bar{I}_2^J$ are the first and second invariants of $\bar{\mathbf{J}}$.

5.4 Peridynamic Model for Spatial Gradient

The previous sections 5.2 and 5.3 finish building nonlocal models to evaluate the Fréchet derivates of solid energy density $\underline{\nabla}\Psi^s$ and volume ratio $\underline{\nabla}J$. In this section, we will continue to build a nonlocal model to the spatial gradient terms $\nabla_{\mathbf{x}}\phi^\alpha$ that appears in (4.21) .

First, we introduce our peridynamic left Cauchy-Green deformation tensor $\bar{\mathbf{B}}$ similarly as how we define our peridynamic right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$ in (5.1) :

$$
\bar{\mathbf{B}}=\left[\int_{\mathcal{H}}\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{Y_iY_j\xi_m\xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right]:\bar{L}_{mnkk},
\tag{5.40}
$$

which can be proved to be identical to the classical left Cauchy-Green deformation tensor $\bar{B}$ when the deformation is homogeneous, i.e. $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle=\mathbf{F}\boldsymbol{\xi}$:

$$
\begin{aligned}
\bar{B}_{ij}=&\left[\int_{\mathcal{H}}\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{F_{ip}\xi_pF_{jq}\xi_q\xi_m\xi_n}{|\boldsymbol{\xi}|^4}\,\mathrm{d}\boldsymbol{\xi}\right]:\bar{L}_{mnkk},\\
=&F_{ip}F_{jq}\bar{K}_{pqmn}L_{mnkk},\\
=&F_{ip}F_{jq}\delta_{pk}\delta_{qk},\\
=&F_{ik}F_{jk},\\
=&B_{ij}.
\end{aligned}
\tag{5.41}
$$

Then, we can use this nonlocal deformation tensor $\bar{\mathbf{B}}$ to pull the spatial gradient $\nabla_{\mathbf{x}}\phi^\alpha$ back to the reference configuration to evaluate it and we propose the following

nonlocal model using indicial notation

$$(\underline{\nabla}_{\mathbf{x}}\phi^{\alpha})_i = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\phi^{\alpha}(\mathbf{X}+\boldsymbol{\xi}) - \phi^{\alpha}(\mathbf{X})\right) Y_j \xi_m \xi_n \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnkk} \cdot \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \tag{5.42}$$

When the deformation is homogeneous and ϕ^{α} is continuous inside $\mathcal{H}$, i.e.

$$\phi^{\alpha}(\mathbf{X}+\boldsymbol{\xi}) = \phi^{\alpha}(\mathbf{X}) + \nabla_{\mathbf{x}}\phi^{\alpha} \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle. \tag{5.43}$$

Substituting (5.43) along with $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi}$ into (5.42) gives

$$\begin{aligned}
(\underline{\nabla}_{\mathbf{x}}\phi^{\alpha})_i &= \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\phi^{\alpha}(\mathbf{X}) + (\nabla_{\mathbf{x}}\phi^{\alpha})_l Y_l - \phi^{\alpha}(\mathbf{X})\right) Y_j \xi_m \xi_n \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnkk} \cdot \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \nabla_{\mathbf{x}}\phi_l \cdot \underline{Y}_l \underline{Y}_j \xi_m \xi_n \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnkk} \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \nabla_{\mathbf{x}}\phi_l \cdot F_{lr}\xi_r F_{js}\xi_s \xi_m \xi_n \mathrm{d}\boldsymbol{\xi} \cdot \bar{L}_{mnkk} \cdot \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \nabla_{\mathbf{x}}\phi_l \cdot F_{lr}F_{js}\bar{K}_{rsmn} \cdot \bar{L}_{mnkk} \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \nabla_{\mathbf{x}}\phi_l \cdot F_{lr}F_{js}\delta_{rs} \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \nabla_{\mathbf{x}}\phi_l \cdot B_{lj} \left(\bar{\mathbf{B}}^{-1}\right)_{ji}, \\
&= \nabla_{\mathbf{x}}\phi_i.
\end{aligned} \tag{5.44}$$

which proves that our nonlocal spatial gradient $\underline{\nabla}\phi^{\alpha}$ is identical to the classical spatial gradient $\nabla_{\mathbf{x}}\phi^{\alpha}$ when the deformation is homogeneous and ϕ^{α} is continuous.

As a final remark, during the previous derivations in this section, ϕ^{α} can be directly replaced by any local function that has dependency on spatial position vector $\mathbf{x}$ in the current configuration, which means the nonlocal model stated as (5.42) is suitable for evaluating the spatial gradient of any local function that has dependency on $\mathbf{x}$ in the current configuration, pulling spatial gradient in the current configuration back to the reference configuration through the peridynamic left Cauchy-Green deformation tensor $\bar{\mathbf{B}}$.

Chapter 6

Correspondence of Peridynamic Theory to Classical Theory for Poroelasticity

Now, we are able to evaluate the three terms $\underline{\nabla}\Psi^s$, $\underline{\nabla}J$ and $\underline{\nabla}_{\mathbf{x}}\phi^\alpha$ nonlocally in our peridynamic momentum equations for poroelasticity (4.21) . Therefore, equations (4.21) along with (5.29) , (5.33) , (5.39) and (5.42) provide a completely nonlocal theory for describing the motion of a porous media. In this chapter, in order to further validate our peridynamic theory for poroelasticity as well as our peridynamic constitutive models, we will prove that the momentum equations in our peridynamic theory (4.21) are identical to the momentum equations derived using continuum mechanics (3.26) in the case that deformation is homogeneous and the geometry of the body together with other physical quantities are continuous.

The peridynamic equations for poroelasticity (4.21) state

$$\begin{aligned}\rho_0^s\ddot{\mathbf{u}}^s =&\rho_0^s\mathbf{G}+\mathbf{H}^s+\int_{\mathcal{H}}\left(p(\mathbf{Q})\phi^s(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-p(\mathbf{X})\phi^s(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\right)d\boldsymbol{\xi}\\&+pJ\underline{\nabla}_{\mathbf{x}}\phi^s-\int_{\mathcal{H}}\underline{\nabla}\Psi^s(\mathbf{Q},\overline{\rho}^s(\mathbf{Q}))\langle-\boldsymbol{\xi}\rangle-\underline{\nabla}\Psi^s(\mathbf{X},\overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\,d\boldsymbol{\xi},\end{aligned}\tag{6.1a}$$

$$\begin{aligned}\rho_0^f\ddot{\mathbf{u}}^f =&\rho_0^f\mathbf{G}+\mathbf{H}^f+\int_{\mathcal{H}}\Big(p(\mathbf{Q})\phi^f(\mathbf{Q})\underline{\nabla}J(\mathbf{Q})\langle-\boldsymbol{\xi}\rangle-p(\mathbf{X})\phi^f(\mathbf{X})\underline{\nabla}J(\mathbf{X})\langle\boldsymbol{\xi}\rangle\Big)\,d\boldsymbol{\xi}\\&+pJ\underline{\nabla}_{\mathbf{x}}\phi^f.\end{aligned}\tag{6.1b}$$

Compare (6.1) with the classical momentum equations for poroelasticity (3.26) which state

$$\rho_0^s \mathbf{a}^s = \rho_0^s \mathbf{G} + \mathbf{H}^s - \phi^s \nabla \cdot \left(p J^s (\mathbf{F}^s)^{-\intercal} \right) + \nabla \cdot \mathbf{P}'^s, \tag{6.2a}$$

$$\rho_0^f \mathbf{a}^f = \rho_0^f \mathbf{G} + \mathbf{H}^f - \phi^f \nabla \cdot \left(p J^f (\mathbf{F}^f)^{-\intercal} \right), \tag{6.2b}$$

and we notice that we only need to prove that the following equations:

$$-\phi^\alpha \nabla_{\mathbf{X}} \left(p J^f (\mathbf{F}^f)^{-\intercal} \right) = \int_{\mathcal{H}} \left(p(\mathbf{Q}) \phi^\alpha(\mathbf{Q}) \underline{\nabla} J(\mathbf{Q}) \langle -\boldsymbol{\xi} \rangle - p(\mathbf{X}) \phi^\alpha(\mathbf{X}) \underline{\nabla} J(\mathbf{X}) \langle \boldsymbol{\xi} \rangle \right) d\boldsymbol{\xi} + p J \underline{\nabla}_{\mathbf{x}} \phi^\alpha, \qquad \alpha = s, f \tag{6.3}$$

$$\nabla \cdot \mathbf{P}'^s = \int_{\mathcal{H}} \underline{\nabla} \Psi^s(\mathbf{Q}, \overline{\rho}^s(\mathbf{Q})) \langle -\boldsymbol{\xi} \rangle - \underline{\nabla} \Psi^s(\mathbf{X}, \overline{\rho}^s(\mathbf{X})) \langle \boldsymbol{\xi} \rangle \, d\boldsymbol{\xi}. \tag{6.4}$$

For the simplicity of the following derivations, use Term A and Term B to denote the peridynamic integrals in (6.3) and (6.4) :

$$\text{Term A} := \int_{\mathcal{H}} \left(p(\mathbf{Q}) \phi^\alpha(\mathbf{Q}) \underline{\nabla} J(\mathbf{Q}) \langle -\boldsymbol{\xi} \rangle - p(\mathbf{X}) \phi^\alpha(\mathbf{X}) \underline{\nabla} J(\mathbf{X}) \langle \boldsymbol{\xi} \rangle \right) d\boldsymbol{\xi}, \tag{6.5}$$

$$\text{Term B} := \int_{\mathcal{H}} \underline{\nabla} \Psi^s(\mathbf{Q}, \overline{\rho}^s(\mathbf{Q})) \langle -\boldsymbol{\xi} \rangle - \underline{\nabla} \Psi^s(\mathbf{X}, \overline{\rho}^s(\mathbf{X})) \langle \boldsymbol{\xi} \rangle \, d\boldsymbol{\xi}. \tag{6.6}$$

When the deformation is homogeneous, and the deformation vector-state $\underline{\mathbf{Y}} \langle \boldsymbol{\xi} \rangle$ can be written as

$$\underline{\mathbf{Y}} \langle \boldsymbol{\xi} \rangle = \mathbf{F} \boldsymbol{\xi}. \tag{6.7}$$

Substituting (6.7) into (5.36) , the peridynamic tensor-state $\bar{\mathbf{J}}$ becomes

$$\bar{\mathbf{J}} = \mathbf{F} \mathbf{F}^\intercal = \mathbf{B},$$

where $\mathbf{B}$ is the classical left Cauchy-Green deformation tensor. Therefore, according to (5.39) , $\underline{\nabla} J$ becomes

$$(\underline{\nabla} J)_i = \underline{\omega} \langle \boldsymbol{\xi} \rangle \frac{\left(I_2^B \delta_{ij} - I_1^B B_{ij} + B_{ik} B_{kj} \right) F_{jl} \xi_l}{K J}, \tag{6.8}$$

where I_1^B and I_2^B are the first and second invariants of the classical deformation

tensor $\mathbf{B}$. If $p\phi^\alpha$ is also assumed to be continuously differentiable, we can have the approximation:

$$p(\mathbf{Q})\phi^\alpha(\mathbf{Q}) = p(\mathbf{X})\phi^\alpha(\mathbf{X}) + \nabla_{\mathbf{X}}(p\phi^\alpha)\cdot\boldsymbol{\xi}. \tag{6.9}$$

Substituting (6.8) and (6.9) into Term A (6.5) using indicial notation gives

$$\begin{aligned}
(\text{Term A})_i &= -\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(p(\mathbf{Q})\phi^\alpha(\mathbf{Q}) - p(\mathbf{X})\phi^\alpha(\mathbf{X})\right) \frac{\left(I_2^B\delta_{ij} - I_1^B B_{ij} + B_{ik}B_{kj}\right)F_{jl}\xi_l}{KJ}\, d\boldsymbol{\xi}, \\
&= -\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\nabla_{\mathbf{X}}(p\phi^\alpha)\cdot\boldsymbol{\xi}\right) \frac{\left(I_2^B\delta_{ij} - I_1^B B_{ij} + B_{ik}B_{kj}\right)F_{jl}\xi_l}{KJ}\, d\boldsymbol{\xi}, \\
&= -\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_m \xi_m \cdot \frac{I_2^B F_{il}\xi_l - I_1^B B_{ij}F_{jl}\xi_l + B_{ik}B_{kj}F_{jl}\xi_l}{KJ}\, d\boldsymbol{\xi}, \\
&= -\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_m \frac{I_2^B F_{il} - I_1^B B_{ij}F_{jl} + B_{ik}B_{kj}F_{jl}}{KJ} \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \xi_m\xi_l d\boldsymbol{\xi}, \\
&= -\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_m \frac{I_2^B F_{il} - I_1^B B_{ij}F_{jl} + B_{ik}B_{kj}F_{jl}}{J}\delta_{ml}, \\
&= -\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_l \frac{I_2^B F_{il} - I_1^B B_{ij}F_{jl} + B_{ik}B_{kj}F_{jl}}{J}.
\end{aligned}$$

Multiplying both sides with $\mathbf{F}^\intercal$, we have

$$\begin{aligned}
F_{im}^\intercal\,(\text{Term A})_i &= -F_{im}\frac{I_2^B F_{il} - I_1^B B_{ij}F_{jl} + B_{ik}B_{kj}F_{jl}}{J}\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_l, \\
&= -\frac{I_2^B F_{im}F_{il} - I_1^B F_{im}F_{ik}F_{jk}F_{jl} + F_{im}F_{in}F_{kn}F_{ks}F_{js}F_{jl}}{J}\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_l, \\
&= -\frac{I_2^B C_{ml} - I_1^B C_{mk}C_{kl} + C_{mn}C_{ns}C_{sl}}{J}\left(\nabla_{\mathbf{X}}(p\phi^\alpha)\right)_l.
\end{aligned}$$

Rewrite this result in tensor form:

$$\mathbf{F}^\intercal\,(\text{Term A}) = -\left(I_2^B\mathbf{C} - I_1^B\mathbf{C}^2 + \mathbf{C}^3\right)\frac{\nabla_{\mathbf{X}}(p\phi^\alpha)}{J}. \tag{6.10}$$

Note that the invariants of $\mathbf{B}$ are equal to the invariatns of $\mathbf{C}$, so we can replace I_1^B

and I_2^B in (6.10) with the invariants of $\mathbf{C}$, I_1^C and I_2^C:

$$\begin{aligned}\mathbf{F}^{\intercal}\,(\text{Term A}) &= -\left(I_2^C\mathbf{C} - I_1^C\mathbf{C}^2 + \mathbf{C}^3\right)\frac{\nabla_{\mathbf{X}}(p\phi^\alpha)}{J},\\ &= -I_3^C\frac{\nabla_{\mathbf{X}}(p\phi^\alpha)}{J},\\ &= -J\nabla_{\mathbf{X}}(p\phi^\alpha),\end{aligned} \tag{6.11}$$

where equation $I_3^C = (\det(\mathbf{F}))^2 = J^2$ and the following characteristic polynomial of tensor $\mathbf{C}$ has been used:

$$\mathbf{C}^3 - I_1\mathbf{C}^2 + I_2\mathbf{C} - I_3 = 0.$$

Therefore, Term A can be written as

$$\begin{aligned}\text{Term A} &= -\mathbf{F}^{-\intercal}J\nabla_{\mathbf{X}}(p\phi^\alpha)\\ &= -\nabla_{\mathbf{X}}(p\phi^\alpha J\mathbf{F}^{-\intercal}).\end{aligned} \tag{6.12}$$

Substituting (6.12) and (5.44) into (6.3) , equation (6.3) becomes

$$\begin{aligned}-\phi^\alpha\nabla_{\mathbf{X}}\left(pJ(\mathbf{F})^{-\intercal}\right) &= -\nabla_{\mathbf{X}}(p\phi^\alpha J\mathbf{F}^{-\intercal}) + pJ\nabla_{\mathbf{x}}\phi^\alpha,\\ &= -\nabla_{\mathbf{X}}(p\phi^\alpha J\mathbf{F}^{-\intercal}) + p\nabla_{\mathbf{X}}(\phi^\alpha J\mathbf{F}^{-\intercal}),\\ &= -p\nabla_{\mathbf{X}}(\phi^\alpha J\mathbf{F}^{-\intercal}) - \phi^\alpha\nabla_{\mathbf{X}}(pJ\mathbf{F}^{-\intercal}) + p\nabla_{\mathbf{X}}(\phi^\alpha J\mathbf{F}^{-\intercal}),\\ &= -\phi^\alpha\nabla_{\mathbf{X}}\left(pJ(\mathbf{F})^{-\intercal}\right),\end{aligned} \tag{6.13}$$

which proves equation (6.3) .

As for Term B, we first rewrite $\underline{\nabla}\Psi^s$ as

$$\underline{\nabla}\Psi^s = \frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}\underline{\nabla}\bar{\mathbf{C}}. \tag{6.14}$$

Letting $m = 1$ in (5.26) gives

$$\underline{\nabla}\bar{\mathbf{C}} = \frac{3}{M}\left(5\,\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}\otimes\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^4} - \underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\mathbf{I}\otimes\underline{\mathbf{Y}}}{|\boldsymbol{\xi}|^2}\right) \tag{6.15}$$

or using indicial notation

$$\underline{\nabla}\bar{C}_{ijk} = \frac{3}{M}\left(5\,\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\xi_i\xi_j Y_k}{|\boldsymbol{\xi}|^4} - \underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\delta_{ij}Y_k}{|\boldsymbol{\xi}|^2}\right). \tag{6.16}$$

When the deformation is homogeneous, equation (5.6) shows that

$$\bar{\mathbf{C}} = \mathbf{C},$$

where $\mathbf{C}$ is the classical right Cauchy-Green deformation gradient. Substituting (6.7) into (6.16) gives

$$\underline{\nabla}\bar{C}_{ijk} = \frac{3}{M}\left(5\,\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\xi_i\xi_j F_{kl}\xi_l}{|\boldsymbol{\xi}|^4} - \underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\delta_{ij}F_{kl}\xi_l}{|\boldsymbol{\xi}|^2}\right). \tag{6.17}$$

And the continuity assumption gives us

$$\frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}(\mathbf{Q}) = \frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}(\mathbf{X}) + \nabla\left(\frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}\right)\cdot\boldsymbol{\xi}. \tag{6.18}$$

Substitute (6.17) and (6.18) into Term B, we have

$$\begin{aligned}
\text{Term B} &= \int_{\mathcal{H}} \underline{\nabla}\Psi^s(\mathbf{Q},\overline{\rho}^s(\mathbf{Q}))\langle-\boldsymbol{\xi}\rangle - \underline{\nabla}\Psi^s(\mathbf{X},\overline{\rho}^s(\mathbf{X}))\langle\boldsymbol{\xi}\rangle\,d\boldsymbol{\xi},\\
&= \int_{\mathcal{H}}\left(\frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}(\mathbf{Q}) - \frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}(\mathbf{X})\right)\underline{\nabla}\bar{\mathbf{C}}\,d\boldsymbol{\xi},\\
&= \int_{\mathcal{H}}\left(\nabla\left(\frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}\right)\cdot\boldsymbol{\xi}\right)\underline{\nabla}\bar{\mathbf{C}}\,d\boldsymbol{\xi},
\end{aligned} \tag{6.19}$$

letting $\mathbf{S} := \frac{\partial\Psi^s}{\partial\bar{\mathbf{C}}}$ and using indicial notation

$$\begin{aligned}
(\text{Term B})_i &= \int_{\mathcal{H}}\left(\frac{\partial S_{jk}}{\partial X_l}\xi_l\right)\underline{\nabla}\bar{C}_{jki}\,d\boldsymbol{\xi},\\
&= \frac{3}{M}\int_{\mathcal{H}}\frac{\partial S_{jk}}{\partial X_l}\left(5\,\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\xi_j\xi_k F_{im}\xi_m}{|\boldsymbol{\xi}|^4} - \underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\delta_{jk}F_{im}\xi_m}{|\boldsymbol{\xi}|^2}\right)\xi_l\,d\boldsymbol{\xi},\\
&= \frac{15}{M}\frac{\partial S_{jk}}{\partial X_l}F_{im}\int_{\mathcal{H}}\underline{\omega}\langle\boldsymbol{\xi}\rangle\frac{\xi_j\xi_k\xi_m\xi_l}{|\boldsymbol{\xi}|^4}\,d\boldsymbol{\xi} - \frac{3}{M}\frac{\partial S_{jj}}{\partial X_l}F_{im}\int_{\mathcal{H}}\underline{\omega}\langle\boldsymbol{\xi}\rangle\xi_m\xi_l/|\boldsymbol{\xi}|^2\,d\boldsymbol{\xi}.
\end{aligned} \tag{6.20}$$

With the assumption that the integral is symmetric and influence function is spherical, we evaluate only the nonzero combinations of indices in the integrals and then the integrations are carried out over a volume of a sphere with radius δ:

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_m \xi_l}{|\boldsymbol{\xi}|^2}\, d\boldsymbol{\xi} = \frac{M}{3}\delta_{ml}, \tag{6.21a}$$

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \frac{\xi_j \xi_k \xi_m \xi_l}{|\boldsymbol{\xi}|^4}\, d\boldsymbol{\xi} = \frac{M}{15}(\delta_{jk}\delta_{ml} + \delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km}). \tag{6.21b}$$

Substituting (6.21) into (6.20) gives

$$\begin{aligned}
(\text{Term B})_i =& \frac{\partial R_{jk}}{\partial X_l} F_{im}(\delta_{jk}\delta_{ml} + \delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km}) - \frac{\partial R_{jj}}{\partial X_l} F_{im}\delta_{ml}, \\
=& \frac{\partial R_{jj}}{\partial X_l} F_{il} + \frac{\partial R_{jk}}{\partial X_k} F_{ij} + \frac{\partial R_{jk}}{\partial X_j} F_{ik} - \frac{\partial R_{jj}}{\partial X_l} F_{il}, \\
=& \frac{\partial R_{jk}}{\partial X_k} F_{ij} + \frac{\partial R_{jk}}{\partial X_j} F_{ik}.
\end{aligned} \tag{6.22}$$

On the other hand, the effective first Piola-Kirchhoff stress $\mathbf{P}'$ defined in (3.20) can be rewritten as

$$\begin{aligned}
P'_{ij} =& \frac{\partial \Psi}{\partial F_{ij}}, \\
=& \frac{\partial \Psi}{\partial C_{mn}} \frac{\partial C_{mn}}{\partial F_{ij}}, \\
=& \frac{\partial \Psi}{\partial C_{mn}} \frac{\partial (F_{km}F_{kn})}{\partial F_{ij}}, \\
=& \frac{\partial \Psi}{\partial C_{mn}} \left(F_{km}\frac{\partial F_{kn}}{\partial F_{ij}} + F_{kn}\frac{\partial F_{km}}{\partial F_{ij}} \right), \\
=& \frac{\partial \Psi}{\partial C_{mn}} \left(F_{km}\delta_{ki}\delta_{nj} + F_{kn}\delta_{ki}\delta_{mj} \right), \\
=& \frac{\partial \Psi}{\partial C_{mj}} F_{im} + \frac{\partial \Psi}{\partial C_{jn}} F_{in}, \\
=& R_{mj}F_{im} + R_{jn}F_{in}.
\end{aligned} \tag{6.23}$$

Hence,

$$\frac{\partial P'_{ij}}{\partial X_j} = \frac{\partial R_{mj}}{\partial X_j} F_{im} + \frac{\partial R_{jn}}{\partial X_j} F_{in}. \tag{6.24}$$

Comparing the results of (6.24) and (6.22) , we have

$$\begin{aligned}(\text{Term B})_i =& \frac{\partial P'_{ij}}{\partial X_j},\\ \text{Term B} =& \nabla \cdot \mathbf{P'},\end{aligned} \tag{6.25}$$

which proves equation (6.4) .

As a result, we prove that the momentum equations in our peridynamic theory (6.1) are identical to the momentum equations derived using continuum mechanics (6.2) if the deformation is homogeneous and the geometry together with other physical quantities are assumed to be continuous, which further validate our peridynamic theory for poroelasticity and our peridynamic constitutive models.

Chapter 7

Conclusions and Recommendations for Future Work

In this work, we first treat the porous media as a binary mixture with solid and fluid and then apply the extended Hamilton principle to describing the motion with finite deformation of the porous media. We further assume that the displacement and geometry of the body are continuous in order to use the theory of continuum mechanics. Then we are able to derive our momentum equations for finite deformation poroelasticity (3.26) where every term is defined in reference configuration with the help of deformation gradient $\mathbf{F}$. To validate our result, we compare our results with Biot's theory, and introduce a new definition for effective stress, which is the virtual force (partial derivative) describing the rate of the change of solid internal energy when changing the displacement field and holding another variable (such as density, volume fraction, pressure and etc.) fixed. We show that if we choose the another variable as pore pressure, we show that our effective stress based on this definition just become Biot's effective stress and the total stress term in our momentum equations becomes effective stress plus Biot's coefficient times pore pressure which is exactly how Biot [1962] and Coussy [1995] express their total stress.

The next part of the thesis, follow the same steps as when we derive momentum equations for finite deformation poroelasticity, but without the continuity assumptions. We also use the extended Hamilton's principle but in a nonlocal way

and we are able to derive a nonlocal momentum equations for poroelasticity (4.21) based on peridynamic theory. However, in order to our peridynamic poroelasticity equations directly applicable to nonlocal numerical simulation of poroelasticity, we still need to find a nonlocal way to evaluate the energy density, volume ratio and spatial gradient. To accomplish that, we introduce our peridynamic constitutive correspondence models for finite deformation.

We first introduce our peridynamic right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$ based on the fourth-order shape tensor $\mathbb{K}$, which can be reduced to the shape tensor $\bar{\mathbf{H}}$ used by [Tupek, 2014] if the symmetry of the integrand and influence function is assumed. The peridynamic deformation tensor $\bar{\mathbf{C}}$ is shown to be identical to the classical right Cauchy-Green deformation tensor. Then similar to the classical Seth-Hill strain family, we also define our peridynamic Seth-Hill strain family $\bar{\mathbf{E}}_{(m)}$ which can avoid the instabilities due to allowing matter interpenetration. So the peridynamic constitutive models for $\bar{\mathbf{C}}$ and $\bar{\mathbf{E}}_{(m)}$ along with the semilinear approximation for energy density allow us to evaluate energy density and its Fréchet derivative nonlocally. Next, we start from the physical interpretation of volume ratio and come up with a nonlocal expression for volume ration using triple integrals, which is impractical to compute directly. Therefore, we introduce a peridynamic tensor $\bar{\mathbf{J}}$ and prove that the determinant of $\bar{\mathbf{J}}$ is equal to the nonlocal expression for volume ratio, which provides an easily-compute way to evaluate volume ratio and its Fréchet derivative nonlocally. Similarly to how we define our peridynamic right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$, we also define our peridynamic left Cauchy-Green deformation tensor $\bar{\mathbf{B}}$ which is able to nonlocally pull spatial gradients from the deformed configuration back to the reference configuration.

Last but not least, as an example of applying our peridynamic constitutive models, we use our peridynamic constitutive models to nonlocally evaluate the solid energy density, the volume ration and the spatial gradient terms in our peridynamic poroelasticity equations, which provides a complete nonlocal theory of describing the motion of porous media as a binary mixture. We show that after substituting our peridynamic constitutive models into peridynamic poroelasticity equations, our nonlocal equations can be reduced to the classical momentum equations derived using continuum mechanics if the deformation is homogeneous and the continuity is assumed, which validate our peridynamic constitutive models.

As for future work, we could extend this nonlocal poroelasticity theory which

treats porous media as a binary mixture to a mixture with N constituents like solid, multiple types of fluids and gases. When the mixture have more that one type of fluids, the energy density of fluids will have extra dependencies on capillary pressures which describe the interface energy between two different types of fluids, which could be really interesting to look into. Besides, the peridynamic poroelasticity equations as well as the peridynamic constitutive models in this thesis are all based on theoretical derivation. Further verification and validation by numerical simulation could be of great help to support the nonlocal theory for poroelasticity and the peridynamic constitutive models for finite deformation.

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